

Fractals

Dietmar Saupe
Institut für Dynamische Systeme
Universität Bremen
W 2800 Bremen-33
Germany

1 Introduction

The greater part of the scientific research of the past consists of the analysis of man-made machines and the physical laws that govern their operation. The success of science relies on the predictability of the underlying experiments. Euclidean geometry — based on lines, circles, etc. — is the tool to describe spatial relations, where differential equations are essential in the study of motion and growth. However, natural shapes such as mountains, clouds or trees do not fit well into this framework. The understanding of these phenomena underwent a fundamental change in the last two decades. *Fractal geometry*, as conceived by Mandelbrot, provides a mathematical model for many of the seemingly complex forms found in nature. One of Mandelbrot's key observations has been, that these forms possess a remarkable statistical invariance under magnification. This may be quantified by a fractal dimension, a number that agrees with our intuitive understanding of dimension but need not be an integer. These ideas may also be applied to time-variant processes. Another important discovery has been that even in very simple nonlinear dynamical systems such as the double pendulum long term predictions are not possible despite exact knowledge of the underlying governing equations. Such systems exhibit behavioral patterns that we can conceive only as erratic or chaotic despite their very simple and deterministic generating mechanisms. Arbitrarily small perturbations of solutions are blown up by such systems until the perturbed solutions have lost all correlation with the original solution. This phenomenon has been termed *sensitive dependence on initial condition* and is the trademark of what became known as *chaos theory*. There is a strong connection between chaos and fractal geometry, namely, as one follows the evolution of the states of a chaotic nonlinear system, it typically leaves a trace in its embedding space which has a very complex geometric structure: this trace is a fractal.

2 Random fractals

Fractal geometric structures exhibit a self-similarity when the distance is changed at which they are viewed. This self-similarity may be either exact or in a statistical

sense. An exact self-similar fractal is the snowflake curve devised by the Swedish mathematician Helge von Koch in 1904, see the construction in figure 1. The curve is self-similar: magnify one quarter of the snowflake curve by a factor of 3 to obtain another complete snowflake curve.

When a self-similar object is given as N copies of itself, each one scaled down by a factor of r , then the *self-similarity dimension* of the object is defined as

$$D = \frac{\log N}{\log 1/r}.$$

This definition assigns a dimension 1 to straight lines and 2 to squares, as expected. Fractals typically have a non-integer dimension. The snowflake curve has a dimension $D = \log 4 / \log 3 \approx 1.262$.

The notion of self-similarity dimension is extended to sets that do not have the *exact* self-similarity. Let A be a set in n -dimensional Euclidean space \mathbf{R}^n , and define $N(r)$ as the minimal number of n -dimensional cubes necessary to cover the set A . Then the (box-counting) *fractal dimension* is

$$D_f(A) = \lim_{r \rightarrow 0} \frac{\log N(r)}{\log 1/r}.$$

This quantity can be estimated from a given data set by drawing a graph of the function $N(r)$ on doubly logarithmic paper. The negative slope of the resulting line fit is an estimate for D_f .

There are other definitions of dimension, e. g. the Hausdorff-Besicovitch dimension, the mass dimension and the correlation dimension.

The mathematical model for a statistically self-similar object is given by *fractional Brownian motion* (fBm). In one dimension fBm is a random process $X(t)$ with Gaussian increments $X(t_2) - X(t_1)$. The variance of these increments is proportional to $|t_2 - t_1|^{2H}$ where $0 < H < 1$. The increments of X are *statistically self-similar with parameter H* . This means, that setting $t_0 = 0$ and $X(t_0) = 0$, the two random functions $X(t)$ and $r^{-H}X(rt)$ are statistically indistinguishable. For a given number X_0 we have that the points t that satisfy $X(t) = X_0$ will constitute a fractal point set, which is statistically self-similar. Its dimension is $D_f = 1 - H$. The graph of $X(t)$ is not self-similar, since we must scale in the t - and X -direction by *different* factors r and $1/r^H$ to obtain statistically equivalent graphs. This form of similarity has been termed *self-affinity*, meaning that properties are invariant under an affine transformation. (Generally, an affine transformation in a Euclidean space is a linear transformation A possibly with a constant offset b such as in $x \mapsto Ax + b$.) The graph of $X(t)$ has a fractal dimension of $2 - H$. Spectral analysis of fBm yields the spectral density $S(f)$ of the process $X(t)$. The density $S(f)$ is proportional to $1/f^\beta$, where the *spectral exponent* β equals $2H + 1$. Thus, β is in the range from 1 to 3.

The generalization of fractional Brownian motion to higher dimensions is a multidimensional process (a *random field*) $X(t_1, t_2, \dots, t_n)$ with the properties analogous to the above. The random field X has *stationary increments* and is *isotropic*, i.e. all points (t_1, t_2, \dots, t_n) and all directions are statistically equivalent. The random fields can also be characterized by their spectral density function or, equivalently, by their autocorrelation function.

Let us consider the case $n = 2$, where $X(t_1, t_2)$ may be plotted as height over the point (t_1, t_2) in the plane. The result is a fractal surface, the graph of X . It is a self-affine fractal whose dimension is $D_f = 3 - H$. The sets of points $\{(t_1, t_2) \text{ satisfying } X(t_1, t_2) = X_0\}$ are collections of curves interpreted as coastlines assuming a water level X_0 . These curves are statistically self-similar with dimension $2 - H$.

The above models describe *uniform fractals*, i. e. fractal properties are essentially global properties. In nonuniform fractals the dimension may require different values. These *multifractals* have received wide attention in the study of aggregation problems.

Another topic of interest here is the *lacunarity* of a fractal. It is a property independent of the fractal dimension but has an effect on the texture or appearance of the fractal and thus is a useful additional parameter for algorithms for the simulation of natural shapes (see below).

3 Algorithms for random fractals

There have been many algorithms developed which generate finite approximations of random fractals. Initially, methods have been judged primarily with respect to the quality of the approximation and raw compute speed, two conflicting goals. More recently, with the advent of increased availability of workstation compute power as well as very high built in graphics capabilities research emphasis is more on flexibility and control of the fractals. For example, local control of the fractal dimension is desirable to model "smooth" valleys surrounded by rough mountains in a landscape scene.

In this section we present a few selected algorithms first for the one-dimensional case and then for two or more dimensions. One of the first and most widely known methods is the *midpoint displacement method*. Assume that values $X(0) = 0$ and $X(1)$ are given. $X(1)$ may be obtained as a sample of a Gaussian random variable of variance σ^2 . The interval $[0, 1]$ is partitioned into two subintervals $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$, and $X(\frac{1}{2})$ is defined as the average of $X(0)$ and $X(1)$ plus a displacement D_1 , i. e.

$$X\left(\frac{1}{2}\right) = \frac{1}{2}(X(0) + X(1)) + D_1 .$$

The displacement D_1 is computed as a sample of a Gaussian random variable with variance Δ_1^2 proportional to $\sigma^2/2^{2H}$. The process is repeated with the two intervals, i. e. , more precisely, we set in this second stage

$$\begin{aligned} X\left(\frac{1}{4}\right) &= \frac{1}{2}(X(0) + X\left(\frac{1}{2}\right)) + D_2 , \\ X\left(\frac{3}{4}\right) &= \frac{1}{2}(X\left(\frac{1}{2}\right) + X(1)) + D_2 , \end{aligned}$$

where D_2 is Gaussian with variance Δ_2^2 proportional to $\sigma^2/(2^2)^{2H}$. Note that the two samples of D_2 in the above formulae may be different. The process is continued with displacements D_n having variances Δ_n^2 proportional to $\sigma^2/(2^n)^{2H}$ in the n -th stage. This method is fast, but lacks mathematical purity since the process X does not have stationary increments for $H \neq \frac{1}{2}$.

One method which improves on the stationarity of the increments of X is called *successive random additions*. Assume that $X(t)$ is already approximated on an

interval at equidistant points with grid size Δt , and let $r > 1$ be a fixed number denoting a reduction factor. In the next step the grid size is reduced to $\Delta t/r$ and values at the new equidistant points are defined by an interpolation procedure (e. g. linear interpolation). Additionally, all values are offset by a sample of a Gaussian random variable with a proper choice of variance. This procedure is repeated until the desired resolution is achieved. E. g. if we start out with just two values of X as in the midpoint displacement algorithm, then in order to obtain an appropriate random fractal with N points, we must exercise n stages of the successive random additions method, where $n \geq \log N / \log r$. The variance Δ_n^2 of the displacement in the n -th such stage of the algorithm must be proportional to $1/r^{2nH}$. The parameter $r > 1$ controls the lacunarity of the fractal. With a large value of r only very few stages are necessary and the lacunarity is especially drastic.

An alternative method is to sum

$$X(t) = \sum_{k=k_0}^{k_1} \frac{\mathcal{S}(r^k t)}{r^{kH}},$$

where $r > 1, 0 < H < 1$ and \mathcal{S} is an auxiliary function similar to the sin and cos functions. E. g. \mathcal{S} may be defined as a smooth interpolant of random data at integer points $t = 0, \pm 1, \pm 2, \dots$. For $k_0 = -\infty, k_1 = \infty$ we obtain a random fractal, whose graph has a fractal dimension $2 - H$ and $r > 1$ determines lacunarity. In practice, the numbers k_0, k_1 are chosen to reflect the upper and lower *crossover scales* of the fractal, i. e. basically $r^{-k_0 H}$ and $r^{-k_1 H}$ will define the largest and the smallest structures seen in the graph of X . This method is a summation of band limited functions, is also called the "rescale-and-add-method", and in this one-dimensional formulation almost the same as the Mandelbrot-Weierstrass function. The parameters r and H determine lacunarity and fractal dimension ($D = 2 - H$) of the graph of $X(t)$. They need not be fixed globally, but may change depending e. g. on t or even $X(t)$.

The generalization of the methods to random fields $X(t_1, t_2)$ is as follows: In the midpoint displacement method we start out with an equilateral triangle and random values of $X(t_1, t_2)$ at the three vertices. Each side is subdivided into two halves and the displacements are done on each side just as in the one-dimensional case. This yields four smaller equilateral triangles with sides half as long. This procedure is iterated until the desired resolution is achieved. It can be modified to operate on squares in place of triangles.

The method of successive random additions is also very easy to implement in the two-dimensional case. One works with grids where the grid sizes are given by $1/r^n$ and multi-linear interpolation may be applied.

In the rescale-and-add-method we set

$$X(t_1, t_2) = \sum_{k=k_0}^{k_1} \frac{\mathcal{S}(r^k t_1, r^k t_2)}{r^{kH}}.$$

Nothing is changed except that the auxiliary function now has two arguments and must be modified accordingly.

Comparing the above methods we get: The midpoint displacement method and its variants are the fastest. However, increments are not stationary and in consequence, one obtains the so called *creasing effect* which disturbs the natural look of

the fractal. For a small extra expense the method of successive random additions offers improved results. The rescale-and-add-method is relatively slow in one dimension but it is superior in three dimensions in which case the other methods suffer from storage problems and time complexity. Moreover, dimension and lacunarity may be changed not only globally, but also locally.

The output of the discussed methods is a two-dimensional array of heights. There are several computer-graphical methods on hand for a rendering. Squares or triangles may be shown as shaded polygons with z-buffer or scanline techniques. When many data are given a floating horizon method can be applied. For most realistic images ray tracing techniques are suitable. Generally it holds that the rendering of fractal surfaces takes more compute time than the generation process for the fractal itself.

4 Deterministic fractals

Random fractals involve an element of chance. In contrast, deterministic fractals are given by means of exact formulae. In this section we consider those deterministic fractals which arise from *discrete dynamical systems*, i. e. the iteration of mappings. These may be derived e. g. from *population dynamics* in biology and yield maps that describe growth of population from one generation to the next. Iteration of the maps simulate the dynamics over longer time periods. Other mappings are motivated by time-variant processes described by differential equations and associated *Poincaré sections*.

The first discovered system of differential equations for which a fractal structure is central consists of the Lorenz equations (1963)

$$\begin{aligned}\dot{x} &= \sigma(y - x) , \\ \dot{y} &= Rx - y - xz , \\ \dot{z} &= xy - bz .\end{aligned}$$

These equations, named after the meteorologist E. Lorenz, were motivated by the problem of weather forecasting and represent a much simplified model of Rayleigh-Bénard convection in fluids. As solutions are followed they tend to a set in 3-space with a complicated fractal structure, a *strange attractor* (see figure 2).

One way to study the dynamics given by a system of three differential equations such as the Lorenz equations consists in reducing the description to a two-dimensional map, called the Poincaré section. A model for the Lorenz system has been suggested by Hénon and Pomeau (1976)

$$\begin{aligned}x_{k+1} &= 1 + y_k - ax_k^2, \quad a = 1.4, \\ y_{k+1} &= bx_k, \quad b = 0.3.\end{aligned}$$

Given an initial point (x_0, y_0) the formula defines a successor point (x_1, y_1) and all following points iteratively. Again, there is a strange attractor with self-similar fractal structure (see figure 3). It is remarkable that important aspects of complex dynamical behavior found in nature can be captured in such simple discrete maps.

A related discrete model is the quadratic mapping

$$z_{k+1} = R_c(z_k) = z_k^2 + c ,$$

where $z_k, k = 0, 1, 2, \dots$ are complex numbers, and c is a complex parameter. This iteration has found widespread interest not only in the scientific community but also among amateur scientists due to its computer graphical potential. It is the iteration procedure which yields the Mandelbrot set

$$M = \left\{ c \in \mathbf{C} \mid \lim_{k \rightarrow \infty} z_k \neq \infty \text{ with } z_0 = c \right\}$$

and the Julia set J_c , which is the minimal completely invariant closed subset of \mathbf{C} . I. e. , we have that $z \in J_c$ if and only if $z^2 + c \in J_c$ (except in the special case $c = 0$ where the Julia set is the unit circle).

The self-similarity of the Julia set is as follows. As in the case of exact self-similar fractals, any small neighborhood of a point in the Julia set can be mapped onto the complete Julia set. However, the necessary similarity mapping is not affine, but *nonlinear*. The fractal dimension of J typically is a noninteger value between 0 and 2. The theory for Julia sets has been carried out not only for the quadratic map but for polynomials and rational maps. E.g. Julia sets naturally arise when a complex polynomial equation $p(z) = 0$ is solved numerically by Newton's method. This amounts to the iteration of $N(z) = z - p(z)/p'(z)$. The roots of the equation $p(z) = 0$ are attractors for the rational map $N(z)$, and the Julia set is the boundary that separates the corresponding basins of attraction. It is the locus of instability: In any arbitrarily small neighborhood of a point in J one finds points that converge to different roots of $p(z)$.

Most phenomena that occur in context with rational maps already appear in the quadratic map $R(z) = z^2 + c$. The Mandelbrot set reflects qualitative aspects for all parameters c of this map, namely it collects all parameters c whose Julia set J_c is a connected set. Outside of the Mandelbrot set corresponding Julia sets are not connected, they are just clouds of points. The Mandelbrot set itself is also a fractal with a certain self-similarity: Any small neighborhood of a boundary point of M contains a complete copy of M . The conjecture that the boundary of M has a dimension equal to 2 has recently been proved.

The computer code to generate the data for an image of the Mandelbrot set is very short, it implements the integer-valued function

$$L(c) = \begin{cases} \ell = \min\{k : |R_c^k(c)| > R_{\max}\} & \text{if } \ell \leq L_{\max} \\ \infty & \text{otherwise} \end{cases}$$

Here $R_c^k(c)$ denotes z_k , the k -th iterate of $z \rightarrow z^2 + c$, starting at $z_0 = c$, L_{\max} is the maximal number of allowed iterations per point, and R_{\max} is a large number (≥ 2). Thus, $L(c)$ is the number of iterations necessary to detect that the critical value c of R_c escapes to ∞ . The computation of $L(c)$ is carried out once per pixel of the image, each pixel representing a small region in complex parameter space. Colors are assigned using a color look-up-table.

$L(c)$ is an integer-valued function. It is possible to define a smooth version by means of the potential function of the Mandelbrot set. For points $c \notin M$ the *potential* is given by

$$G(c) = \lim_{k \rightarrow \infty} \frac{\log |R_c^k(c)|}{2^k} .$$

The expression in the limit converges rapidly once $|R_c^k(c)|$ is large.

The values of $G(c)$ can be used in an image again in connection with a color look-up-table or they are interpreted as a third spatial coordinate, i. e. height, see the color images.

An alternative measurement for points $c \notin M$ is the distance $d(c, M)$ of c to M , which can be estimated according to

$$\frac{\sinh G(c)}{2 e^{G(c)} |G'(c)|} < d(c, M) < \frac{2 \sinh G(c)}{|G'(c)|}.$$

This estimate gives rise to different pictures, see picture 12, and also to a new fast algorithm to compute the Mandelbrot set: Once the lower estimate on the left side has been computed for a point $c \notin M$, then a disk with that radius can be eliminated from further computation, since it is guaranteed that this disk does not intersect M . A speed-up factor of 10 or higher may result, depending on the region of the image and the resolution.

The algorithms for the computation of color images of Julia sets are very similar to the above, for details see the literature.

5 Applications

Fractals are found in nature almost everywhere, in the very large scales of clusters of galaxies down to the microcosmos of molecular particles. Although the mere description of these shapes and forms in terms of fractals does not explain anything it is clear that the fractal "dialect" is the appropriate language and, thus, it will evolve to become a lasting element of science.

Obviously, wherever nature is to be *simulated*, fractals are of value. Landscapes and clouds were two of the first natural phenomena discussed in the computer graphics community. Research is now incorporating effects such as erosion of a fractal landscape allowing rivers and river networks to be included in the long list of simulated phenomena.

Fractal geometry applied to spatial relationships provides one way to achieve visual complexity. Another method used in computer graphics is called *solid texturing*, which is usually given as a functional model. E. g. in the image 8 showing a fractal planet, the geometry is very simple and not fractal: it is merely a sphere. But the texturing function which determines the coloring is constructed using the laws of random fractals. As demonstrated there is a balance between geometry and texture. No visual complexity is lost if one of the two is relaxed while the other one is enhanced.

One of the most important aspects of fractals is that their visual complexity is very high in contrast to their complexity which is measured in terms of the length of the shortest computer program that can generate them and which is very small. This is called *data base amplification* and is important where large, detailed data bases are too big or too costly to be used, e. g. in flight or driving simulators, or telecommunication.

A systematic analysis of images is possible using self-similarity and affine maps in an approach using *partitioned iterated function systems*. One greyscale or color image, e. g. a rasterized photography, a synthetic computer graphics image, or a

transmitted satellite image, may be approximated and encoded as an object that is invariant under a collection of maps. These affine maps may be stored in much less memory as compared to the original image. In turn, these affine maps may again be used to reconstruct the image — or at least a close approximation of it. More specifically, fractal image compression works as follows. The original image is partitioned into a set of non-overlapping, so-called range blocks. For each range block a matching domain block is searched for. A domain block is an image block of twice the linear size. The match is obtained first by downsampling to the range block size, and then by adjusting gain and brightness parameters to achieve the best possible approximation of the range block in the L_2 sense. This parameter optimization is a least squares operation. When all range blocks are replaced by their transformed domain block approximations, the so-called image collage is obtained. It is the job of the encoder to make this collage as close to the original as possible. The fractal code of the image contains the partitioning information and a list of gain and brightness parameters along with corresponding domain block addresses. The decoder cannot reconstruct the collage since it does not have access to the original domain blocks that the encoder has selected. Instead an approximation of the collage is obtained by iterating the collage operator. Starting out with an arbitrary image as input, its collage as specified by the code is constructed as output. Then the process is iterated, taking the last output image as the input image for the next iteration. This iteration converges, provided certain contractivity constraints are met. There are many important issues in fractal image compression that have been researched, e.g., the choice of the partitioning of the image, the quantization scheme of the parameters, and the reduction of the computational complexity. Recently hybrid methods, combining fractal coding with the wavelet or discrete cosine transformation have provided very good coding results.

There are numerous other applications of fractal geometry in image processing and pattern recognition. Two of them are: automatic segmentation of images based on fractal dimension, lacunarity, etc. which is useful in differentiating objects in an image such as man-made structures, forest and sky, and optimization of camouflage methods based on fractal analysis of the surroundings.

A method to generate fractal shapes that grow in space is based on *Lindenmayer systems*. Objects are represented as strings of symbols which are generated from an *axiom* (initial short string) and a set of *production rules* which are applied recursively to the symbols of the axiom and the resulting strings. The geometric representation of these strings is obtained through *turtle graphics*. Classic fractal curves such as the snowflake curve, Hilbert's space filling curve, the Sierpinski gasket etc. are easily and compactly formulated as L-systems. The main application is the modelling of growth and form of trees, bushes and plants. These results stem from an interdisciplinary effort in computer science and biology.

These above mentioned applications are relevant to computer science and engineering. Perhaps the most important uses are in the physical sciences. One area of applications is given by *percolation processes* describing a fluid spreading randomly through a medium, which supplies a network of narrow pores and channels for the fluid flow. This includes seepage of (possibly contaminated) water through the cracks and fractures of rock formation.

Another application is diffusion limited aggregation in which 'sticky' particles

move about randomly and eventually attach to a cluster of particles. In this process the cluster grows into a fractal characterized by its fractal dimension. There is a strong coincidence with forms generated by viscous fingering in porous media, which is an important issue for oil recovery.

It is not a surprise that molecular surfaces are fractal, and thus, fractal geometry will prove useful in chemical engineering (catalysis) and other areas where surface/volume relations are crucial, e. g. wetting and powder technology.

The deterministic fractals typically arise from dynamical systems which are motivated by numerous models for natural phenomena. E. g. the production of red blood cells in the bone marrow may be modelled using a delay differential equation, which exhibits chaotic attractors which in turn may be interpreted as serious irregularity in the red blood cell concentration in patients with leukemia.

The Mandelbrot set plays a central role as a *paradigm* for the transition from order to chaos in such models. As the parameter c is decreased along the real axis, the system $z \mapsto z^2 + c$ undergoes a series of *period doubling bifurcations*, a phenomenon that has also been discovered in several physical laboratory experiments. Moreover, it has been revealed that the Mandelbrot set is in some sense a *universal object*. It appears also in the study of rational and so-called polynomial-like maps.

As a side effect due to their beauty Julia sets and the Mandelbrot set have provided inspiration for computer art and serve as a pleasing demonstration object on computer trade shows. Another unforeseen but maybe relevant side effect of the colorful Mandelbrot set images is that they convey a hint at the beauty that lies within mathematics. Thus, they supply motivation to regain the widely lost assertion that mathematics is a worthwhile and, in fact, important part of the human culture.

References

- [1] Barnsley, M. F., *Fractals Everywhere*, (Academic Press, 1988)
- [2] Fisher, Y., *Fractal Image Compression — Theory and Application*, (Springer-Verlag, New York, 1995)
- [3] Gleick, J., *Chaos - Making a New Science*, (Viking, New York, 1987)
- [4] Hao B.-L., *Chaos II*, (World Scientific, Singapore, 1990)
- [5] Mandelbrot, B. B., *The Fractal Geometry of Nature*, (W.H.Freeman and Co., New York, 1982)
- [6] Peitgen, H.-O. and Richter, P. H., *The Beauty of Fractals*, (Springer-Verlag, Heidelberg, 1986)
- [7] Peitgen, H.-O. and Saupe, D. (eds), *The Science of Fractal Images*, (Springer-Verlag, New York, 1988)
- [8] Peitgen, H.-O., Jürgens, H., and Saupe, D., *Fractals for the Classroom, Part One and Part Two*, (Springer-Verlag, New York, 1991)

- [9] Prusinkiewicz, P., and Lindenmayer, A., *The Algorithmic Beauty of Plants*, (Springer-Verlag, New York, 1990)

Black and white figures captions

Figure 1 Construction of the von Koch snowflake curve. The interval $[0,1]$ is given initially (not shown here). In each stage (going from bottom to top) line segments are replaced by the generator curve consisting of four lines as shown in the bottom curve (stage 1). As stages are added the total length of the curve tends to infinity although the curve is confined to a finite region.

Figure 2 The Lorenz attractor for parameters $R = 28$, $\sigma = 10$, $b = 8/3$.

Figure 3 The Hénon chaotic attractor in two enlargements, computed from a run of 100,000 points. The regions shown are $[0, 1] \times [0, 0.3]$ (left) and $[0.7, 0.8] \times [0.15, 0.18]$ (right). The small square in the left figure corresponds to the enlargement on the right. The self-similar structures are clearly visible.

Figure 4 The Julia set corresponding to Newtons's method for the solution of $z^3 - 1 = 0$. The thick points indicate the three roots of the equation.

Figure 5 The Mandelbrot set with some equipotential lines.

Color figures captions

Figure 6 Electrostatic potential around a small satellite Mandelbrot set. The Julia set belonging to the parameter c from the center of this Mandelbrot set is pictured on the (Riemann) sphere in the background. Cover picture of *The Beauty of Fractals* [6]. [©1986 H. Jürgens, H.-O. Peitgen, D. Saupe]

Figure 7 The potential function near the Mandelbrot set. The rendering of the sky is by means of random fractals (method of successive random additions). Cover picture of *The Science of Fractal Images* [8]. [©1988 H. Jürgens, H.-O. Peitgen, D. Saupe]

Figure 8 Fractal planet generated with the rescale-and-add-method. The fractal dimension depends on the latitude: Near the equator the dimension of the coast lines is close to 1.0, whereas near the poles it is close to 2.0. The planet is rendered as a perfect sphere. However, the texturing function is based on the random fractal (pseudo) height and, for the polar caps, also depending on latitude. [©1988 H. Jürgens, H.-O. Peitgen, D. Saupe]

Figure 9 Random fractal in three variables rendered as a cloud with a fractal moon generated via the random cuts method and a background motivated by the filigrees of the Mandelbrot set. From *The Science of Fractal Images* [8]. [©1988 R. Voss]

Figure 10 Random fractal landscape with haze. [©1989 F. K. Musgrave, B. Mandelbrot]

Figure 11 Fractal mountain scene with tree grown by L-systems. [©1989 F. K. Musgrave, C. Kolb, B. Mandelbrot, P. Prusinkiewicz]

Figure 12 Spheres filling the exterior of the Mandelbrot set. They are computed using the distance estimate formula. [©1988 H. Jürgens, H.-O. Peitgen, D. Saupe]

Index keys

successive random additions
fractals, random
fractals, deterministic
fractal geometry
fractal dimension
dynamical system
sensitive dependence on initial condition
chaos theory
self-similarity
von Koch snowflake curve
fractional Brownian motion
self-affinity
spectral density
height field
multifractal
lacunarity
midpoint displacement
crossover scales
rescale-and-add-method
Mandelbrot-Weierstraß function creasing effect
Poincaré section
Lorenz equations
strange attractor
Mandelbrot set
Julia set
Newton's method
basin of attraction
potential function
solid texturing
fractal planet
data base amplification
iterated function system
fractal image compression
image segmentation
camouflage method
Lindenmayer system
axiom
production rules
Hilbert curve
Sierpinski gasket
percolation
diffusion limited aggegration