Acta Applicandae Mathematicae **13** (1988), 59–80.

Discrete Versus Continuous Newton's Method: A Case Study

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(Received: 17 October 1986)

Abstract. We consider the damped Newton's method $N_h(z) = z - hp(z)/p'(z)$, $0 < h < 1$ for polynomials $p(z)$ with complex coefficients. For the usual Newton's method ($h = 1$) and polynomials $p(z)$, it is known that the method may fail to converge to a root of p and rather leads to an attractive periodic cycle. $N_h(z)$ may be interpreted as an Euler step for the differential equation $\dot{z} = -p(z)/p'(z)$ with step size h. In contrast to the possible failure of Newton's method, we have that for almost all initial conditions to the differential equation that the solutions converge to a root of p. We show that this property generally carries over to Newton's method $N_h(z)$ only for certain nondegenerate polynomials and for sufficiently small step sizes $h > 0$. Further we discuss the damped Newton's method applied to the family of polynomials of degree 3.

AMS subject classifications (1980). 65H05, 30D05, 58F12.

Key words. Relaxed Newton's method, basins of attraction.

1. Introduction

For a large class of computational problems a zero of a map $f: \mathbf{X} \to \mathbf{X}$, where $\mathbf{X}$ typically is the Euclidean space $\mathbf{R}^n$ or the complex space $\mathbf{C}^n$, has to be found. Given that f is differentiable, Newton's method is a first candidate for the numerical solution: Let $x_0 \in \mathbf{X}$ be an initial guess for a zero of f.

Compute

$$x_{k+1} = N(x_k) = x_k - Df(x_k)^{-1} f(x_k), \quad k = 0, 1, 2, \ldots$$

Typically, the sequence $(x_k)_{k=0,1,\ldots}$ will converge to a zero x^* of f, and locally convergence is of second order if x^* is a zero of f such that $Df(x^*)$ is nonsingular. In this paper we will discusss global properties of Newton's method. In particular, we are interested in the set of initial guesses $x_0 \in \mathbf{X}$ that lead to a failure of the method. First, it is clear that the method fails if the initial guess generates a sequence $(x_k)_{k=0,1,\ldots}$ which eventually becomes undefined, i.e., $x_0 \in C_f$, where

$$C_f = \{x_0 \in \mathbf{X} \mid Df(N^k(x_0)) \text{ is singular for some } k\}.$$

The initial guesses $x_0 \in \mathbf{X}$ which lead to a converging sequence of Newton iterates form the basins of attraction of the zeros x^* of f:

$$B_f(x^*) = \{x_0 \in \mathbf{X} \mid \lim_{k\to\infty} N^k(x_0) \to x^*\}.$$

Clearly, the basins of attraction $B_f(x^*)$ are open sets.

We address the question whether Newton's method and its damped version $N_h(x) = x - hDf(x)^{-1}f(x)$ converges almost everywhere for complex polynomials f, or

$$\mathbf{X} = \mathrm{cl} \bigcup_{x^* \in f^{-1}(0)} B_f(x^*).$$

In this case, the iterated map is a rational map and the theory of Julia and Fatou (see the surveys in [2, 4] and [8, 9] including many color illustrations) applies. In the case of a real map $f: \mathbf{R}^n \to \mathbf{R}^n$ there are some features of the iteration of N that are reminiscent of the iteration of rational maps in $\mathbf{C}$ but also there are some striking new phenomena (see [7, 8]).

2. The Continuous Newton's Method

Before we go into the details of the discrete Newton's method we present its continuous counterpart. Newton's iterate $N(x)$ may be interpreted as an Euler step to the differential equation

$$\dot{x} = -Df(x)^{-1}f(x)$$

for $x \in \mathbf{X}$ such that $Df(x)$ is nonsingular. Here the basin of attraction of a zero x^* of f is given by

$$B_f(x^*) = \{x \in \mathbf{X} \mid \Phi_t(x) \to x^* \text{ as } t \to \infty\}$$

where Φ_t denotes the 1-parameter group of solutions of the differential equation. Again we can ask whether

$$\mathbf{X} = \mathrm{cl} \bigcup_{x^* \in f^{-1}(0)} B_f(x^*).$$

For polynomials $f: \mathbf{C} \to \mathbf{C}$ this question is settled. There we have

$$\dot{z} = -\frac{f(z)}{f'(z)}. \tag{1}$$

In order to remove the singularities which occur at the critical points of f we rescale the equation to

$$\dot{z} = -f(z)\overline{f'(z)}. \tag{2}$$

For the solutions $z(t)$ of (2) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} f(z(t)) = |f'(z(t))|^2 f(z(t)).$$

Thus, if we write the polar coordinates of $z \in \mathbf{C}$ as $z = r(z)\,\mathrm{e}^{i \arg(z)}$ we have that

$$\arg(f(z(t)) \equiv \text{const}$$

along solutions $z(t)$ of (1) and (2) and that $|f(z(t))|$ is a monotonically decreasing function. For this reason the preimages of the rays $\{r\,\mathrm{e}^{i\alpha} \mid r > 0\}$ w.r.t. the polynomial f are of interest:

$$C_\alpha = \{z \in \mathbf{C} \mid \arg(f(z)) = \alpha \text{ and } f(z) \neq 0\}.$$

If C_α contains no critical points of f, then C_α is a collection of disjoint trajectories of solutions of (2). These solutions all tend to a zero of f. If C_α contains some critical points of f, then C_α is a collection of these critical points and joint trajectories of solutions of (2) which tend to one of these critical points or to a zero of f. Therefore, we define the critical line of f as the set

$$S_f = \mathrm{cl}\{z \in \mathbf{C} \mid \lim_{t \to \infty} \Phi_t(z) \to z_c\,,\, f'(z_c) = 0, f(z_c) \neq 0\}$$

of trajectories which are stable manifolds of those critical points z_c of f which are not roots of f. As the counterpart of the critical line, we define the connecting line as the set

$$L_f = \mathrm{cl}\{z \in \mathbf{C} \mid \lim_{t \to \infty} \Phi_{-t}(z) \to z_c\,,\, f'(z_c) = 0, f(z_c) \neq 0\}$$

of trajectories that are unstable manifolds of those critical points. L_f is an oriented planar graph where the trajectories correspond to edges and critical points and zeros of f correspond to vertices.

THEOREM 1 (Smale, Braess). *Let $f: \mathbf{C} \to \mathbf{C}$ be a polynomial with disjoint roots. If there are no two critical values of f lying on the same ray, then the following hold for the connecting line L_f:*

(1) *L_f is connected.*
(2) *There are exactly two edges ending at each critical point of f. There are no edges between critical points or between zeros of f.*
(3) *L_f has no cycles.*

For the critical line we have:

THEOREM 2 (Braess). *Let $f: \mathbf{C} \to \mathbf{C}$ be a polynomial. The critical line of f is the boundary of the basins of attraction of the roots of f with respect to $\dot{z} = -f(z)\overline{f'(z)}$.*

Let us remark that the critical line intersects the connecting line orthogonally

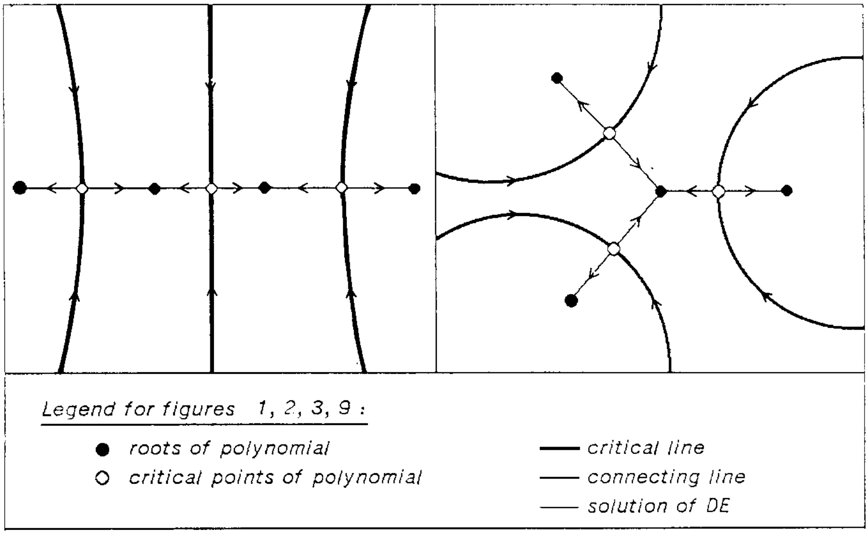

Fig. 1. Examples of critical lines and connecting lines for polynomials of degree 4. No two critical values are on the same ray. The critical line splits **C** into 4 basins of attraction.

at simple critical points of f which are not also roots of f. To see this we note that the expansion of the right side of (2) $\dot{z} = -f(z)\overline{f'(z)}$ at such a critical point is

$$\dot{z} = -\overline{(z-z_0)f''(z_0)f(z_0)} + O(|z-z_0|^2).$$

Setting

$$w = \frac{z - z_0}{\sqrt{\overline{f''(z_0)f(z_0)}}}$$

we have for w

$$\dot{w} = -|\overline{f''(z_0)}f(z_0)|\overline{w} + O(|w|^2).$$

The linearized equation for $\dot{w}$ has the stable manifold **R** and the unstable manifold i**R**.

For a detailed analysis of the facts outlined above see the papers of Braess [3], Jongen, Jonker and Twilt [6] and Smale [10]. The point relevant in the following is that the continuous Newton's method (2) converges almost everywhere, and that the critical line separates the basins of attraction of the roots of f.

3. The Discrete Newton's Method

Newton's method $N(z) = z - f(z)/f'(z)$ for polynomials $f\colon \mathbf{C} \to \mathbf{C}$ generally does not have the property that it converges for almost all initial points $z \in \mathbf{C}$.

However, if $f: \mathbf{R} \to \mathbf{R}$ is a polynomial with only real roots, the following theorem of Barna ([1]) holds.

THEOREM 3 (Barna). *If $f: \mathbf{R} \to \mathbf{R}$ is a polynomial which has only real roots, then Newton's method $x_{k+1} = x_k - f(x_k)/f'(x_k)$, $k = 0, 1, 2, \ldots$ converges to a root of f for all initial points $x_0 \in R \setminus K$ where K is a Cantor set of real numbers.*

If we allow complex roots of the polynomial f it is easy to construct (real) polynomials such that Newton's method exhibits attractive periodic cycles. Following [10] we derive conditions on the coefficients a_k, $k = 0, \ldots, n$ in

$$f(z) = a_0 + a_1 x \cdots + a_n x^n, \quad a_n \neq 0$$

such that

$$N(0) = 1,\ N(1) = 0,\ N'(0) = 0,\ N'(1) \neq \infty.$$

Then we have that $N^2(0) = 0$ and $(N^2)'(0) = 0$, and there is an open neighborhood of 0 such that all points of it converge to the periodic cycle $0, 1, 0, 1, \ldots$.

We have that $N(0) = 1$ if $a_0 + a_1 = 0$ and $a_0, a_1 \neq 0$. Since $N'(0) = 2a_0a_2/a_1^2$ the

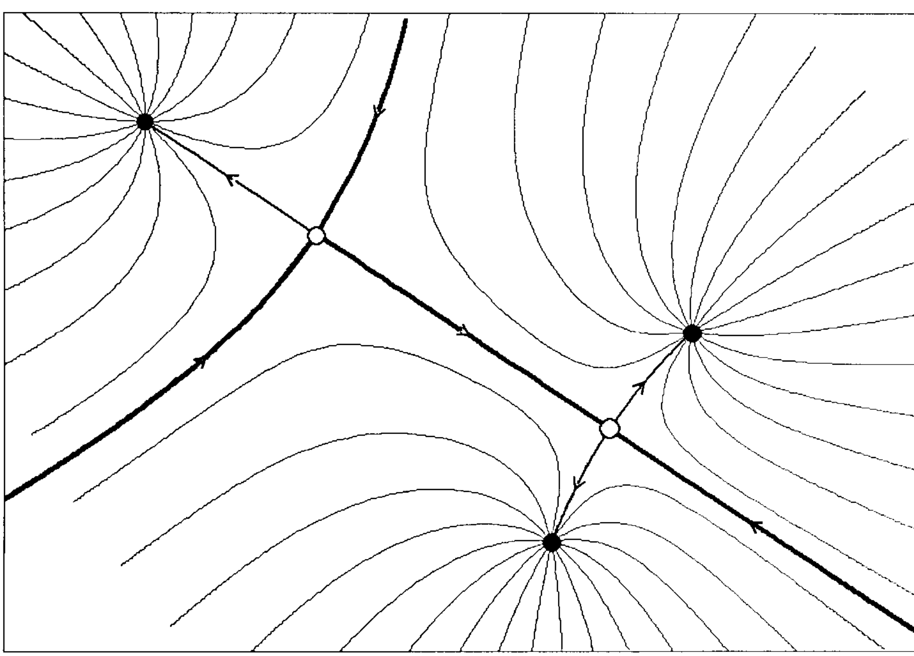

Fig. 2. The degenerate case: For the two critical points z_{c1} and z_{c2} we have $\arg(f(z_{c1})) = \arg(f(z_{c2}))$. Note that the solution of (2) which connects the critical points is shared by the connecting line and the critical line. The critical line splits $\mathbf{C}$ into 3 basins of attraction.

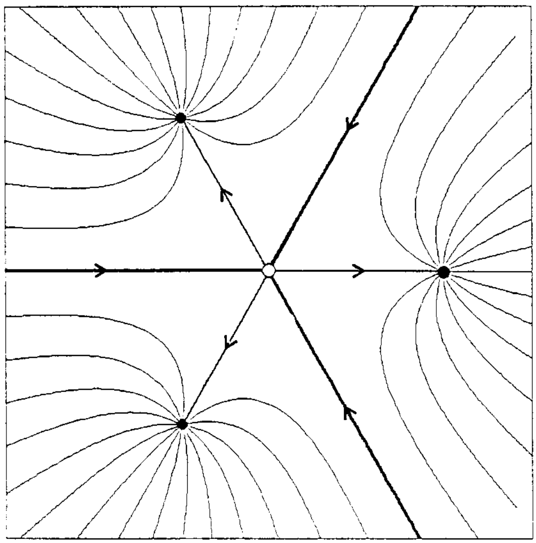

Fig. 3. The degenerate case $f(z) = z^3 - 1$.

condition $N'(0) = 0$ is fulfilled if $a_2 = 0$. $N(1) = 0$ is satisfied if

$$-a_0 + 2a_3 + 3a_4 + \cdots + (n-1)a_n \neq 0,$$

Clearly these conditions can be satisfied if only the degree of f is greater than 2, e.g., the polynomial

$$f(z) = \tfrac{1}{2} z^3 - z + 1$$

is of the desired type. Newton's method started sufficiently close to 0 or 1 will be cycling asymptotically between 0 and 1.

If a polynomial f is given it is possible to check the existence of attractors of Newton's method that are not roots of f. The following theorem of Fatou provides the necessary tool to decide whether there are other attractors.

THEOREM 4 (Fatou). *Let R: $\mathbf{C} \to \mathbf{C}$ be a rational function of degree ≥ 2. If R has an attractor, then its immediate basin of attraction, i.e. the connected component of the basin which contains the attractor, has at least one critical point of R.*

This theorem bounds the number of attractors that may arise through iteration of R: If n is the degree of R then the number of critical points of R, thus also the number of attractors, is bounded by $2(n-1)$.

The critical points of Newton's method N are the zeros of

$$N'(z) = \frac{f(z)f''(z)}{(f'(z))^2}$$

which are the roots of f and of f'' that are not also roots of f'. The roots of f are not interesting as critical points, because they are fixed points of N. However, the critical points which are roots of f'' are free critical points in the sense that their iteration may lead to a periodic attractor of N. And if there is a periodic attractor of N (of period 2 or greater) then Fatou's theorem assures us that at least one of the free critical points must tend to the attractor.

This observation facilitates a comprehensive study of Newton's method for *all* polynomials p of degree 3. It turns out that it suffices to only consider a one-parameter family of polynomials of degree 3. We first remark that an affine change of coordinates

$$g\colon \mathbf{C} \to \mathbf{C},\ g(z) = az + b$$

is in order: Let

$$N(z) = z - \frac{p(z)}{p'(z)}, \qquad q(z) = p(g(z)), \qquad \tilde{N}(z) = z - \frac{q(z)}{q'(z)}.$$

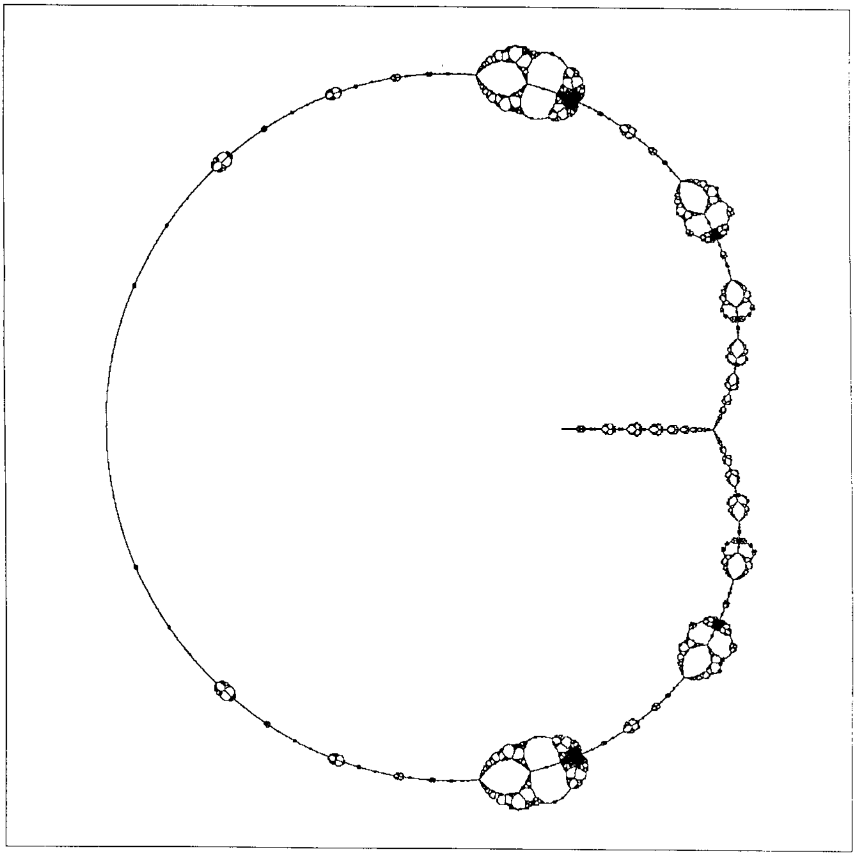

Fig. 4. Where the free critical point does not tend to a root of $p_\lambda(z) = z^3 + (\lambda - 1)z - \lambda$. Pictured is the plane of parameters λ, $-2.75 < \mathrm{Re}\,\lambda < 1.75$, $-2.25 < \mathrm{Im}\,\lambda < 2.25$.

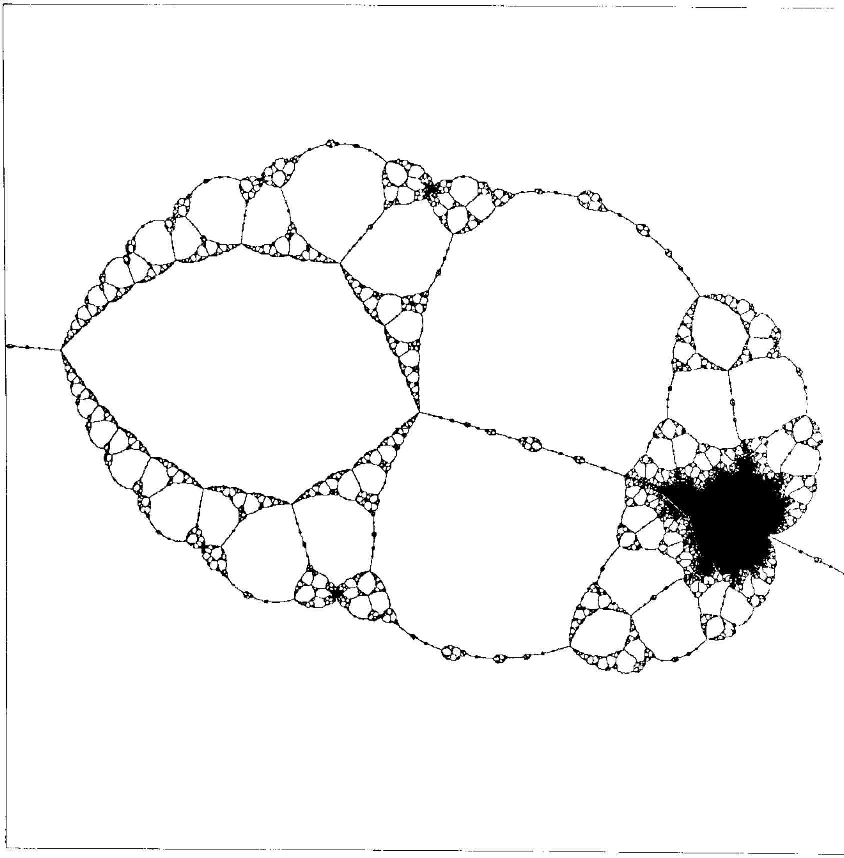

Fig. 5. Detail of Figure 4 with Mandelbrot-like set, $-0.2 < \operatorname{Re} \lambda < 0.4$, $1.4 < \operatorname{Im} \lambda < 2.0$.

Then it is easy to see that $\tilde{N} = g^{-1}Ng$ and, thus, the properties of N are in a one-to-one correspondence with the properties of $\tilde{N}$.

Let z_0, z_1, z_2 denote the roots of p. If they are identical, then the affine change of coordinates $g(z) = a(z - z_0)$ yields the case $q(z) = z^3$. In the other case we find (possibly after rearranging the order of the roots) a transformation $g(z) = az + b$ such that

$$g(z_0) = 1, \qquad g(\tfrac{1}{2}(z_1 + z_2)) = -\tfrac{1}{2}.$$

Then $q(z) = p(g(z))$ will be a polynomial which has one root at 1 and all roots sum up to 0. Thus, it suffices to study Newton's method for

$$p_\lambda(z) = (z - 1)(z^2 + z + \lambda),$$

i.e.

$$p_\lambda(z) = z^3 + (\lambda - 1)z - \lambda$$

where $\lambda \in \mathbf{C}$ is a parameter. Newton's method applied to such a polynomial has

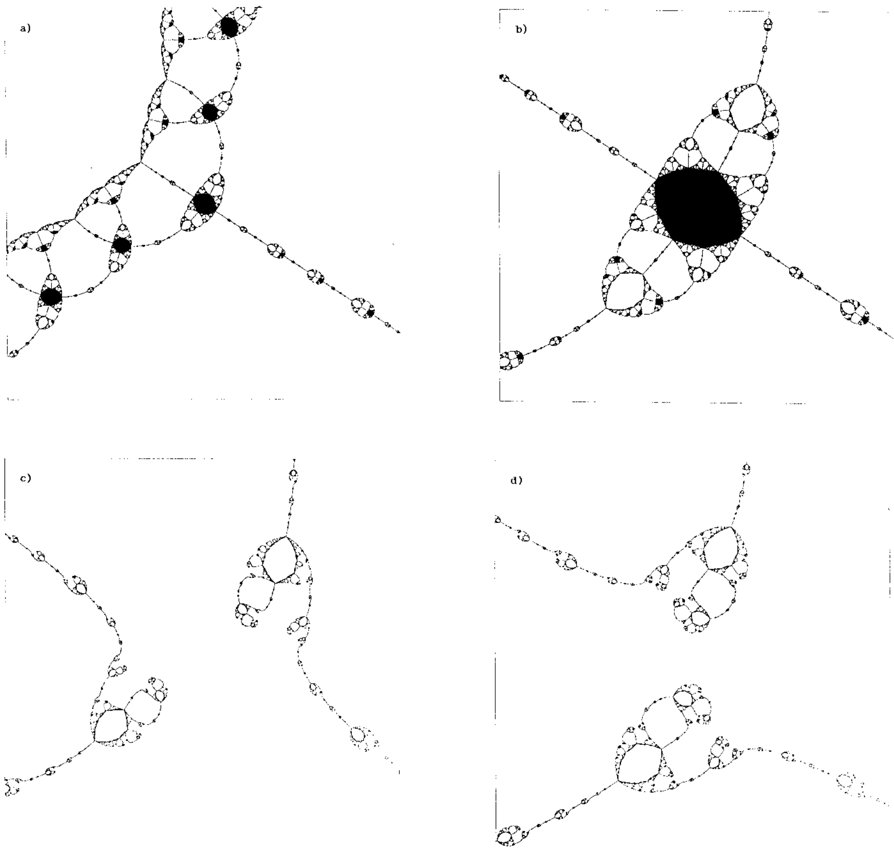

Fig. 6. Basins of attraction of Newton's method for $p_\lambda(z) = z^3 + (\lambda - 1)z - \lambda$. Shown are the boundaries of the basins of attraction of the roots of p and, if it exists (i.e., in (a) and (b)), the basin of attraction of a periodic attractor. The figures cover the region $\{x + iy \in \mathbf{C} \mid -2 \leq x, y \leq 2\}$, except for figure (b) which is a closeup of (a). The parameters are: (a) and (b) $\lambda = (0.3158, 1.6348)$, (c) $\lambda = (0.2842, 1.4713)$, (d) $\lambda = (0.3473, 1.7983)$.

one free critical point which is 0. Therefore, the study of Newton's method applied to polynomials of degree 3 (aside from the trivial case $p(z) = z^3$) reduces to the study of the fate of the free critical point 0 for all parameters $\lambda \in \mathbf{C}$. In this way one obtains a morphology of Newton's method, and this program has been carried out in [5] (and extended in [11] to Schroeder rational iteration maps). The result is summarized in Figure 4. The black regions indicate parameters for which the free critical point does not tend to a root of the polynomial. In the Mandelbrot-like set in Figure 5, a close-up of Figure 4, we have parameters λ yielding periodic attractors of periods $2, 4, 6, \ldots$. In Figure 6 a few of the corresponding basins of attraction are shown.

The algorithm used for the graphical display differs from the one used in [5]: For each pixel the free critical point for *four* parameters λ corresponding to the corners of the pixel (Figures 4 and 5) respectively four initial points z (Figure 6) were tested. If in all cases convergence to the same root of p was achieved, then we assumed that for all parameters resp. initial values covered by the pixel we have convergence to that root and, thus, the color of the pixel was left white. In the other case, or if all four iterations were asymptotically periodic (with period 2 or greater) then the color of the pixel was set to black. In this scheme we compute of course only the iterations for parameters resp. initial points lying on a grid where the pixels correspond to the regions between four neighboring grid points. Thus, for the resolution used for the above figures of 1024 by 1024 we have computed 1025^2 iteration sequences.

4. The Damped Newton's Method

We now study the damped Newton's method

$$N_h(z) = z - h\frac{p(z)}{p'(z)}, \quad 0 \leq h \leq 1$$

for polynomials $p(z)$ of degree 3, i.e., we again set

$$p_\lambda(z) = z^3 + (\lambda - 1)z - \lambda, \quad \lambda \in \mathbf{C}.$$

As $h \to 0$ $N_h(z)$ viewed as a dynamical system is expected to approximate the system given by the differential equation (2) $\dot{z} = -f(z)\overline{f'(z)}$. We ask if there is a cuttoff value h^* such that N_h for $0 < h < h^*$ shares the property with (2) that convergence to a root of p occurs for almost all initial points $z \in \mathbf{C}$. In order to carry out an experiment based on Fatou's theorem along the lines of [5] and the last section, we now have to consider all critical points of N_h which are given as solutions of the equation

$$(1-h)(p'_\lambda(z))^2 + hp_\lambda(z)p''_\lambda(z) = 0.$$

The polynomial on the left side is of degree 4, so there are 4 critical points of N_h and in contrast to the already discussed case $h = 1$ we do not know a priori to which attractor some of the critical points tend. Of course, we have that the 3 roots of p are attractors for N_h for $0 < h < 1$ (in fact $0 < h < 2$ is sufficient), and they each have at least one of the critical points in their basin of attraction. Therefore, no other attractor besides the roots may exist, if one of the roots attracts *two* of the 4 critical points.

In the computer experiments shown in Figure 7 we picture the parameters λ for which each root of p contains only *one* of the critical points.

We observe that as $h \to 0$ the cardiod-shaped curve and its spike on the real line in parameter space persists (see Figure 8) and that the Mandelbrot-like sets and their characteristic neighborhoods shrink in size and move on the cardiod-shaped curve and its spike to the point $(1, 0)$ which corresponds to the polynomial $p(z) = z^3 - 1$. This indicates that the cardiod curve and the spike carry some

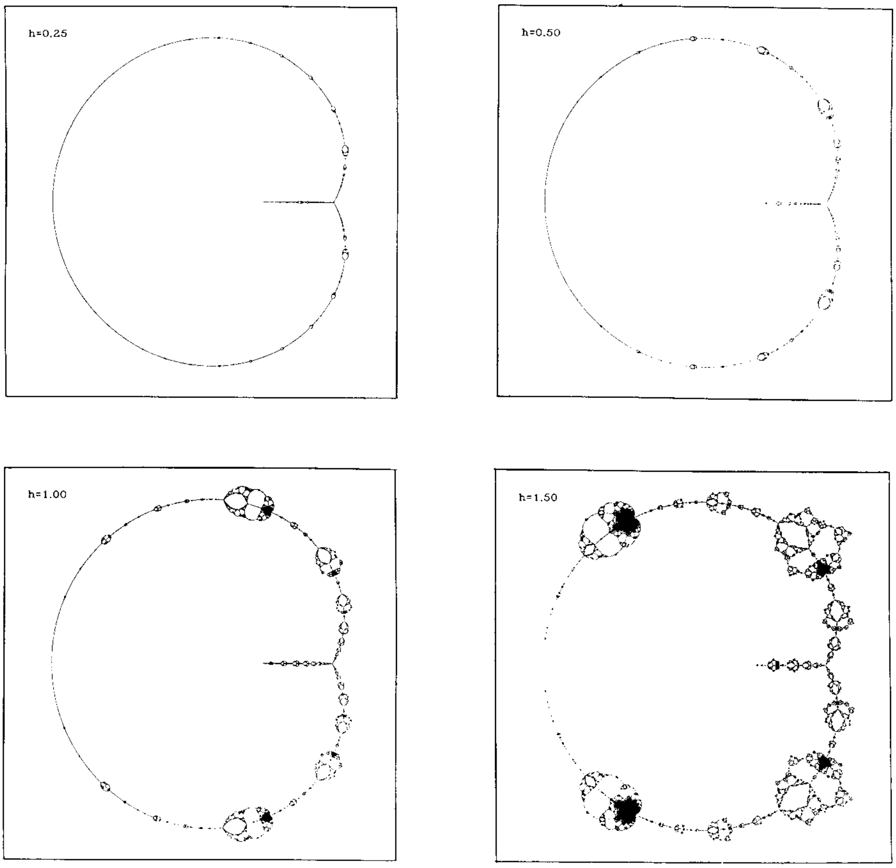

Fig. 7. Where one of the critical points of N_h does not tend to a root of $p_\lambda(z) = z^3 + (\lambda - 1)z - \lambda$. Pictured is the plane of parameters $\lambda, -2.75 < \text{Re}\,\lambda < 1.75$, $-2.25 < \text{Im}\,\lambda < 2.25$.

characteristic property w.r.t. the differential equation (1) or (2). From our computer experiment we see (although this information is not contained in the figures here) that as λ crosses the cardiod curve the root which has two of the critical points of N_h in its basin of attraction loses one of them and another root then attracts two critical points. The corresponding process for the differential equation (2) is outlined in Figure 9.

The qualitative change of the connecting and the critical line occurs precisely at the parameters λ where the two critical values of p_λ lie on the same ray:

$$\arg p_\lambda(z_{c_1}) = \arg p_\lambda(z_{c_2}), \qquad p'_\lambda(z_{c_1}) = p'_\lambda(z_{c_2}) = 0.$$

Thus, we conjecture that the cardiod curve and its real spike are given by the parameters λ that satisfy the above property. Before we compute these parameters we note that for all other parameters λ and sufficiently small step sizes

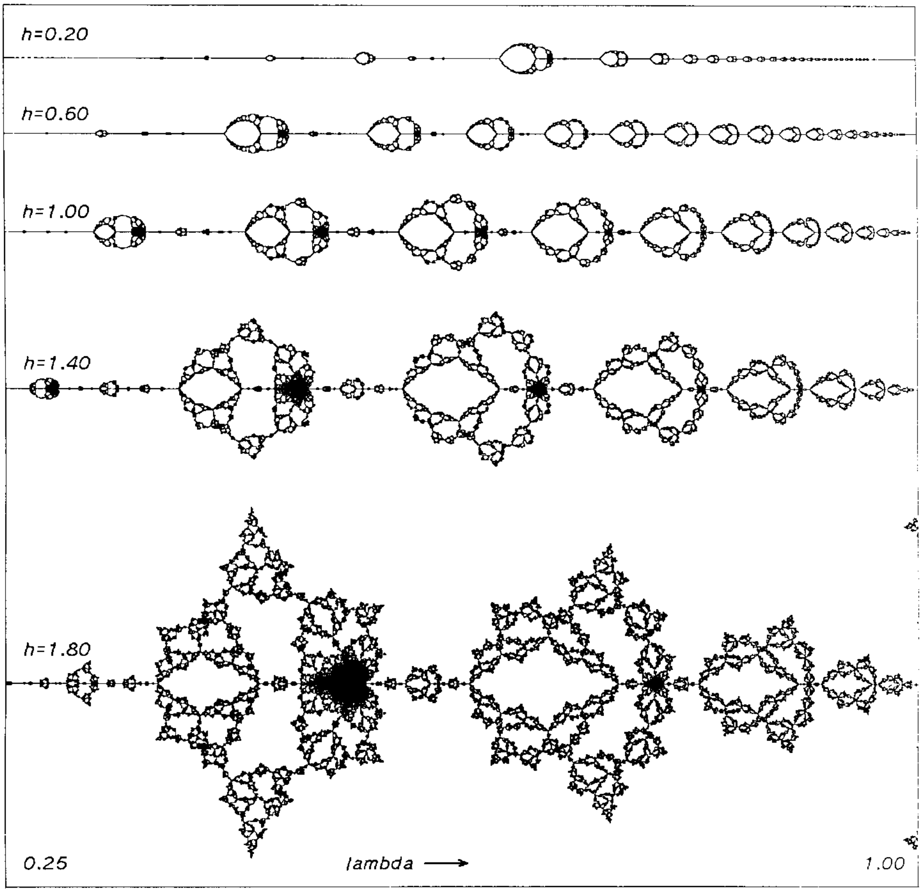

Fig. 8. The real spike around $\frac{1}{4} < \lambda < 1$ for various step sizes h in N_h.

$h > 0$ we apparently have that Newton's method N_h cannot exhibit attractors other than the roots of p_λ. This is a special case of our following conjecture.

CONJECTURE 5. *Let p: $\mathbf{C} \to \mathbf{C}$ be a complex polynomial with disjoint roots and assume that no two critical values of p lie on the same ray. Then, for sufficiently small $h > 0$, the damped Newton's method $N_h(z) = z - hp(z)/p'(z)$ has no other attractors than the roots of p.*

In a proof of the conjecture applying Fatou's theorem, one would have to show that all critical points of N_h, $h > 0$ sufficiently small, are attracted by the roots of p. Since the Julia set of N_h will tend to the critical line as $h \to 0$ we cannot expect the convergence of the critical points of N_h to the roots of p if these critical points lie on the critical line of p. Therefore, the next proposition which states that the critical points approach the critical line orthogonally as $h \to 0$, is a partial proof of the conjecture 5.

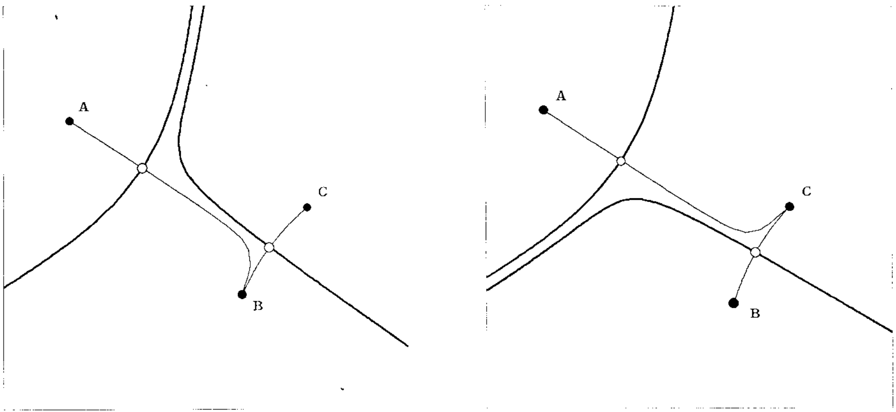

Fig. 9. A small change in λ causes a qualitative change of the connecting line: On the left root B is in the center whereas on the right root C is in the center. The parameters are $\lambda = (0.2842, 1.4713)$ (left), and $\lambda = (0.3473, 1.7983)$ (right), compare Figures 5c,d.

PROPOSITION 6. *Let $p\colon \mathbf{C} \to \mathbf{C}$ be a complex polynomial of degree n with disjoint roots and assume that no two critical values of p lie on the same ray. Let*

$$G = \{z \in \mathbf{C} \mid N_h'(z) = 0 \text{ for some } 0 < h \leqslant 1\}$$

G is a collection of $2n-2$ smooth curves which intersect the critical line of p at the critical points of p orthogonally.

Proof. The critical points of N_h are the solutions of

$$(1-h)(p'(z))^2 + hp(z)p''(z) = 0. \tag{3}$$

If $\deg p = n$, then there are $2n-2$ critical points counting multiplicities. As h approaches 0 the critical points of N_h tend to the critical points of p. More precisely, each critical point of p is the limit of two critical points of N_h as $h \to 0$. Recall that in the nondegenerate case (no two critical values of p on the same ray) we have that the critical points of p are just the intersection points of the $n-1$ components of the critical line of p with the connecting line of p. We show that as $h \to 0$ the critical points of N_h approach these intersections in the same manner as the connecting line, i.e., if $z(h)$ denotes a solution branch of (3) in G then $p(z(h))$ tends to a critical value $z(0)$ of p and

$$\frac{\mathrm{d}}{\mathrm{d}h} p(z(h))|_{h=0} = -cp(z(0))$$

where c is a positive constant. Therefore, the branch $z(h)$ in G intersects the critical line orthogonally at $z(0)$.

By the chain rule and implicit differentiation we obtain

$$2(1-h)p'p''z' - p'p' + pp'' + h(pp'''z' + p'p''z') = 0 \tag{4}$$

where $p = p(z)$ and $z = z(h)$. Suppose that z' is bounded as $h \to 0$. Since $p'(z(0)) = 0$ we obtain from (4) that $p(z(0))p''(z(0)) = 0$. If $p(z(0)) = 0$ we have a double root of p at the critical point $z(0)$ in contradiction to the hypothesis of the proposition. If $p''(z(0)) = 0$, then there is a double critical point of p at $z(0)$ which also is a contradiction. Therefore, z' is unbounded and $|z'(h)| \to \infty$ as $h \to 0$. We rewrite (4)

$$2(1-h)p'' \frac{\mathrm{d}p}{\mathrm{d}h} - p'p' + pp'' + h \frac{\mathrm{d}p}{\mathrm{d}h}\left(\frac{pp'''}{p'} + p''\right) = 0 \tag{5}$$

and observe, that with $p''(z(0)) \neq 0$, $|z'(h)| \to \infty$ as $h \to 0$ and l'Hospital's rule we have

$$\lim_{h \to 0} \frac{h}{p'(z(h))} = \lim_{h \to 0} \frac{1}{p''(z(h))z'(h)} = 0.$$

From this and (5) we finally obtain

$$\frac{\mathrm{d}}{\mathrm{d}h} p(z(h))|_{h=0} = -\frac{1}{2} p(z(0)).$$ □

5. Degeneracy and Symmetry

We now return to the discussion of the one-parameter family of degree 3 polynomials $p_\lambda(z) = z^3 + (\lambda - 1)z - \lambda$. In the next proposition we state that the polynomials not covered by Conjecture 5 and Proposition 6 are precisely those whose parameters λ lie on the cardiod curve and its real spike discussed in the last section.

PROPOSITION 7. *The set of parameters $\lambda \in \mathbf{C}$ such that*

$$p_\lambda(z) = z^3 + (\lambda - 1)z - \lambda$$

has two critical values on the same ray $\{r\,\mathrm{e}^{i\alpha} \mid r > 0\}$ or a multiple root is given by the union of the real interval $[\frac{1}{4}, 1]$ and the (algebraic) curve

$$\lambda(\phi) = 2\cos\left(\frac{\pi - \phi}{3}\right)\mathrm{e}^{i\phi}$$

with $0 \leq \phi < 2\pi$.

Proof. p_λ has double roots for $\lambda = -2$ and $\lambda = \frac{1}{4}$. The two critical points and their values are

$$w_{1,2} = \pm\sqrt{\frac{1}{3}(1-\lambda)}, \qquad p_\lambda(w_{1,2}) = \pm 2\left(\frac{1-\lambda}{3}\right)^{3/2} - \lambda.$$

First assume that $\lambda \in \mathbf{R}$. Then

$$\arg p_\lambda(w_1) = \arg p_\lambda(w_2)$$

implies that

$$\lambda \leq 1 \quad \text{and} \quad 2\left(\frac{1-\lambda}{3}\right) < |\lambda|.$$

A short algebraic manipulation shows that the second condition is equivalent to $4(\lambda+2)^2(\lambda-\frac{1}{4})>0$. Thus, the real parameters λ satisfying the hypothesis of the proposition are given by $\lambda=-2$ and $\frac{1}{4}\leq\lambda\leq 1$. We have that the point $\lambda=-2$ is the point on the curve $\lambda(\phi)$ for $\phi=\pi$.

If λ is complex we must have $\arg\lambda = \arg -(1-\lambda)^{3/2}$ for $0<\arg\lambda<\pi$ and $\arg\lambda=\arg(1-\lambda)^{3/2}$ for $\pi<\arg\lambda<2\pi$. Additionally

$$\left|2\left(\frac{1-\lambda}{3}\right)^{3/2}\right| < |\lambda| \tag{6}$$

must hold. We give a geometric argument for the case $0<\arg\lambda<\pi$, the other case being symmetric. If $\phi=\arg\lambda$ and ψ denotes the angle between λ and $\lambda-1$ (see Figure 10), i.e., $\psi=\arg\lambda-\arg(1-\lambda)$, then $\arg(1-\lambda)=\phi+\psi+\pi$ and we conclude that $(3(\phi+\psi+\pi))/2=\phi+2\pi$. Therefore, $\psi=(\pi-\phi)/3$ and using the law of sines it follows that

$$\frac{1}{\sin\psi}=\frac{|\lambda|}{\sin(\pi-\phi-\psi)}$$

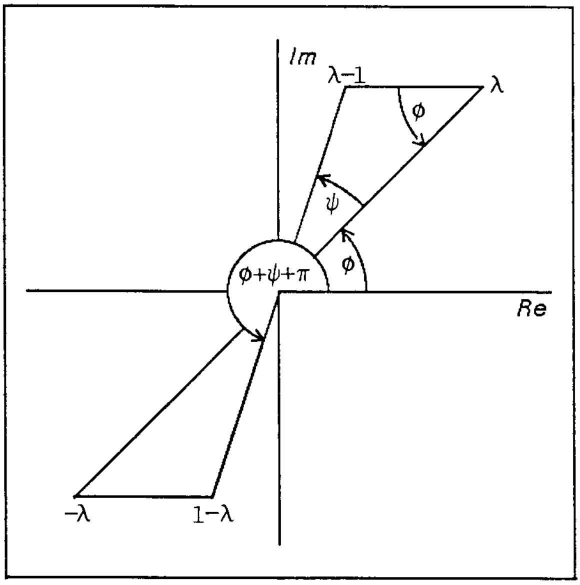

Fig. 10. To the proof of Proposition 7.

or

$$|\lambda| \frac{\sin\left(2\dfrac{\pi-\phi}{3}\right)}{\sin\dfrac{\pi-\phi}{3}} = 2\cos\frac{\pi-\phi}{3}.$$

Therefore,

$$\lambda(\phi) = 2\cos\left(\frac{\pi-\phi}{3}\right) e^{i\phi}.$$

The law of sines also yields

$$|1-\lambda| = \frac{\sin\phi}{\sin\dfrac{\pi-\phi}{3}}.$$

Thus, the condition (6) is equivalent to

$$2\cos\left(\frac{\pi-\phi}{3}\right) > \frac{2}{3\sqrt{3}}\left[\frac{\sin\phi}{\sin\dfrac{\pi-\phi}{3}}\right]^{3/2}.$$

Setting $s = \sin((\pi-\phi)/3)$ and writing $\sin\phi$ in terms of s we see that we have to show

$$4(1-s^2) > \frac{4}{27}\frac{(3s-4s^3)^3}{s^3}$$

which is equivalent to $s^2(8s^2-9)^2 > 0$. □

Figures 4, 5 and 7 indicate that the damped Newton's method applied to the one-parameter family p_λ carries some inherent symmetry properties. The first obvious symmetry arises when we substitute the roots

$$1,\ z_{1,2} = \tfrac{1}{2} \pm \sqrt{\tfrac{1}{4}-\lambda}$$

of p_λ by the conjugates 1, $\overline{z_1}$, and $\overline{z_2}$ which are the roots of $p_{\bar\lambda}(z) = z^3 + (\bar\lambda - 1)z - \bar\lambda$.

There is another more intricate symmetry present. Consider the triangle in $\mathbf{C}$ given by the vertices 1, z_1, z_2, the roots of p_λ. Then, since $\frac{1}{2}(z_1+z_2) = -\frac{1}{2}$ we have that the origin is the intersection of the three bisectors of the triangle and it divides these lines with the ratio 1:2 (see Figure 11).

These geometric relations of the roots will persist if we divide all roots by either z_1 or z_2. Thus, 1, $1/z_1$, z_2/z_1 and also 1, $1/z_2$, z_1/z_2 are roots of a polynomial $p_{\tilde\lambda}$ for some parameters $\tilde\lambda$. Since the parameter λ in p_λ is equal to the product of the roots which are not equal to 1 we have $\tilde\lambda = z_2/z_1^2$ or $\tilde\lambda = z_1/z_2^2$. Thus, we obtain

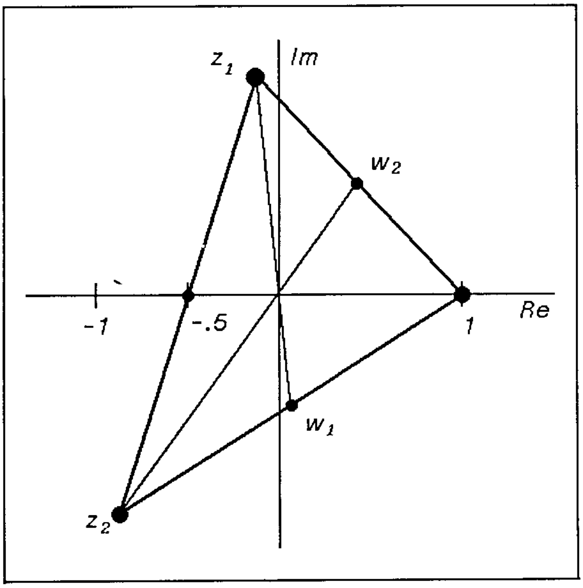

Fig. 11. Root configuration for $p_\lambda(z) = z^3 + (\lambda - 1)z - \lambda$. We have $\frac{1}{2}(z_1 + z_2) = -\frac{1}{2}$, $w_1 = -\frac{1}{2}z_1 = \frac{1}{2}(z_2 + 1)$, and $w_2 = -\frac{1}{2}z_2 = \frac{1}{2}(z_1 + 1)$.

LEMMA 8. *Newton's method N_h applied to $p_\lambda(z) = z^3 + (\lambda - 1)z - \lambda$ is conjugate to the method applied to $p_{\tilde{\lambda}}$ where*

$$\tilde{\lambda} = \frac{\lambda}{(-\frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda})^3} = \frac{1}{\lambda^2}(-\tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} - \lambda})^3 \tag{7}$$

The transformation $\lambda \to \tilde{\lambda}$ has three fixed points 0, 1 and -2. The intervals $(-\infty, -2)$, $(-2, 0)$ and $(0, \frac{1}{4})$ are equivalent in the sense of the above lemma. Of special interest are the parameters $\tilde{\lambda}$ which are associated via (7) with real parameters $\lambda \geq \frac{1}{4}$. They form two parametrized algebraic curves.

LEMMA 9. *Let*

$$A = \left\{ \lambda(t) = \frac{1}{t^2}(-\tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} - t})^3,\ t \geq \tfrac{1}{4} \right\}$$

$$B = \{x + iy \in \mathbf{C} \,|\, (2x^2 + 2y^2 - 3)^2 - 9 + 8x = 0\}.$$

Then $A = B$.

Proof. Let $\lambda(t) \in A$. Then

$$x = \operatorname{Re} \lambda(t) = \frac{1}{2t^2}(3t - 1),$$

$$y = \operatorname{Im} \lambda(t) = \pm \frac{1-t}{t^2}\sqrt{t - \frac{1}{4}},$$

$$x^2 + y^2 = |\lambda(t)|^2 = \frac{1}{t},$$

and it is easy to show that $x + iy \in B$. If $x + iy \in B$, then the assumption $x^2 + y^2 > 4$ implies $x < -2$ and $q(x) = (2x^2 - 3)^2 - 9 + 8x \leqslant 0$. However, since $q(-2) = 0$ and $q'(x) < 0$ for $x \leqslant 2$ we arrive at the contradiction $q(x) > 0$ for $x < -2$. Thus, B is contained in a disk of radius 2 around the origin. For $x + iy \in B$ we set $t = 1/(x^2 + y^2)$. Then $t \geqslant \frac{1}{4}$ and it is easy to check that $\lambda(t) = x + iy$. □

The curve of degenerate parameters $\lambda(\phi) = 2\cos((\pi - \phi)/3)\, e^{i\phi}$ in Proposition 7 can be seen to be a part of the algebraic curve B

$$(2x^2 + 2y^2 - 3)^2 - 9 + 8x = 0$$

as follows. Setting $\lambda = x + iy$ we have for the real part of λ

$$x = -2\cos\left(\frac{\pi - \phi}{3}\right)\cos\phi = -8\cos^4\frac{\pi - \phi}{3} + 6\cos^2\frac{\pi - \phi}{3}$$

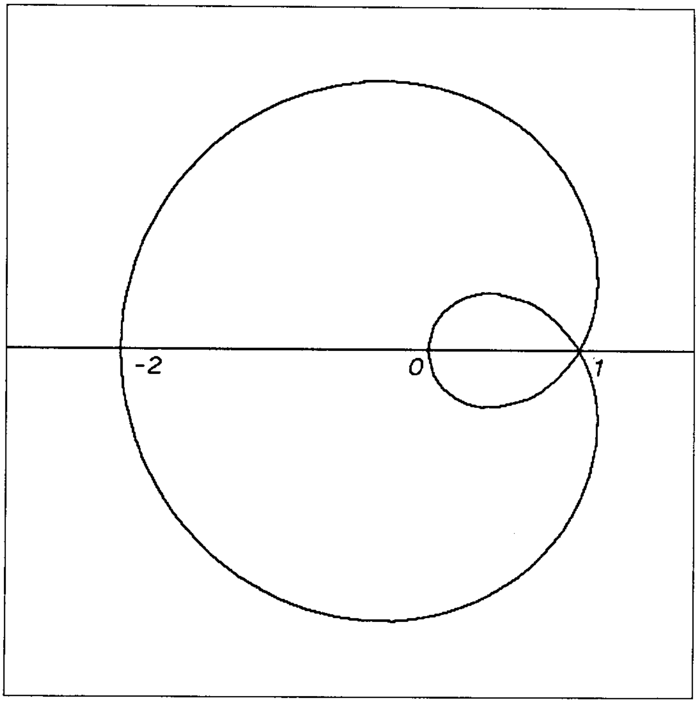

Fig. 12. Six equivalent regions of the complex parameter plane which are separated by the real axis and the algebraic curve $(2x^2 + 2y^2 - 3)^2 - 9 + 8x = 0$.

or

$$\cos^2\frac{\pi-\phi}{3}=\frac{1}{8}(3\pm\sqrt{9-8x}).$$

Since $|\lambda|^2=4\cos^2((\pi-\phi)/3)$ we get $(2x^2+2y^2-3)^2-9+8x=0$.

Since $|\lambda(\phi)|=2\cos((\pi-\phi)/3)\in[1,2]$ we have that $\lambda(\phi)$ (from Proposition 7) is equal to $\lambda(t)$ (from Lemma 8) if

$$t=\frac{1}{4\cos^2\dfrac{\pi-\phi}{3}}\in[\tfrac{1}{4},1].$$

Overall, we reach the following conclusion (see Figure 12): The algebraic curve $(2x^2+2y-3)^2-9+8x=0$ and the real axis divide the complex plane into 6 regions, which are equivalent for the study of Newton's method N_h applied to $p_\lambda(z)=z^3+(\lambda-1)z-\lambda$. Moreover, the set of parameters λ which are degenerate in the sense of Proposition 7 are given by the outer part of the algebraic curve and the spike $[\frac{1}{4},1]$.

6. Newton's Method for Degenerate Polynomials

In Section 4 we have noted that the damped Newton method N_h applied to a polynomial which is not degenerate in the sense that none of its critical values lie on the same ray will not have attractors other than the roots of the polynomial if we only choose the step size $h>0$ sufficiently small. In this section we present experimental evidence that this cannot be true for degenerate polynomials (of degree 3). As a consequence of Proposition 7 and Lemma 8 we have that it suffices to study the degenerate polynomials of degree 3 which are given by

$$p_\lambda(z)=z^3+(\lambda-1)z-\lambda,\quad \tfrac{1}{4}\leqslant\lambda\leqslant 1.$$

The computer experiment we propose is the following: For each choice of $\frac{1}{4}\leqslant\lambda\leqslant 1$ and $0\leqslant h\leqslant 1$ compute the four critical points of N_h applied to p_λ, their orbits under iteration of N_h and determine whether the iteration carries all four critical points to the roots of p_λ. We can reduce the workload of this experiment considerably using the following observation.

LEMMA 10. *Let $N_h(z)=hp_\lambda(z)/p_\lambda'(z)$, $0<h<1$ with $p_\lambda(z)=z^3+(\lambda-1)\,z-\lambda$, $\frac{1}{4}<\lambda<1$. Then N_h has two complex and two real critical points. If z_l and z_r denote the two real critical points, then*

$$z_l<\sqrt{\frac{1-\lambda}{3}}<z_r.$$

Moreover, z_r is in the basin of attraction of the root 1 *of p_λ w.r.t. N_h.*

This lemma implies that since p_λ is a real polynomial having two complex roots the two complex critical points must be attracted by the complex roots. The third critical point, z_r, is attracted by the root 1. Thus, z_l is the only free critical point, i.e., if there is an attractor other than the roots of p_λ, then it must also attract the critical point z_l. Therefore, in our computer experiment we can restrict to the iteration of the real rational map N_h with real starting point z_l.

Proof. p_λ is a real polynomial having the root 1 and a pair of complex conjugate roots. N_h has four critical points which are zeros of

$$q(z) = (1-h)(p'_\lambda(z))^2 + hp_\lambda(z)p''_\lambda(z).$$

Due to the theorem of Fatou we know that two of the critical points much be in the basins of attraction of the two complex roots of p_λ. These critical points cannot be real because the real line is invariant under N_h. Thus, N_h has at least two complex critical points. The other two critical points must be real as seen from

$$q(z) = (9-3h)z^4 + 6(\lambda - 1)z^2 - 6h\lambda z + (1-h)(\lambda - 1)^2,$$

$$q\left(\sqrt{\frac{1-\lambda}{3}}\right) = -\frac{4h}{3}(1-\lambda)^2 - 6h\lambda\sqrt{\frac{1-\lambda}{3}} < 0.$$

Moreover, if z_l and z_r denote these real critical points it follows that

$$z_l < \sqrt{\frac{1-\lambda}{3}} < z_r.$$

To show that z_r is in the basin of attraction of the root 1 we note that $p_\lambda(x)$ is strictly monotonically increasing and concave up for

$$x > \sqrt{\frac{1-\lambda}{3}}.$$

Therefore we have

$$x < N_h(x) \quad \text{for} \quad \sqrt{\frac{1-\lambda}{3}} < x < 1 \quad \text{and} \quad 1 < N_h(x) < x \quad \text{for } x > 1$$

from which the claim readily follows. □

The result of the experiment is shown in Figure 13 and summarized in the following conjecture.

CONJECTURE 11. *For λ given such that $\frac{1}{4} < \lambda < 1$ and $h^* > 0$ there is a step size $h < h^*$ such that N_h applied to $p_\lambda(z) = z^3 + (\lambda - 1)z - \lambda$ reveals some periodic attractor different from the roots of p_λ. The period of this attractor must be unbounded as $h^* \to 0$. More generally, if p is a polynomial with two critical values on the same ray, then the same conclusion holds.*

Finally, we have a partial result related to our conjecture stating that for any

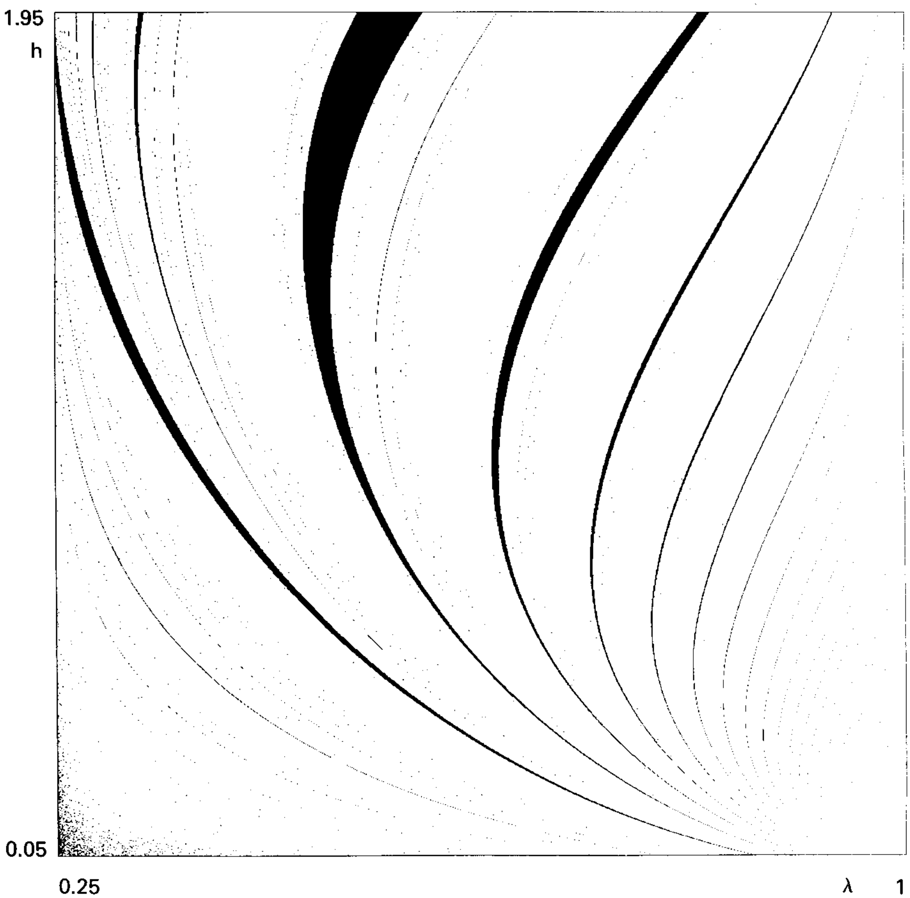

Fig. 13. Computer experiment on Newton's method N_h, $0<h<2$ for degenerate polynomials $p_\lambda(z)=z^3+(\lambda-1)z-\lambda$, $\frac{1}{4}<\lambda<1$. Horizontally λ varies from $\frac{1}{4}$ to 1, vertically h goes from 0.05 to 1.95. Points in black denote choices of λ and h for which the free critical point z_l is not attracted by a root of p_λ.

$h>0$ there is a polynomial for which N_h has a period 2 attractor. The proof is a simple generalization of the case $h=1$ discussed in [10] and in Section 3.

PROPOSITION 12. *For* $0<h\leqslant 1$, $N_h(z)=z-hp(z)/p'(z)$ *there exists a polynomial* $p\colon \mathbf{C}\to\mathbf{C}$ *such that* N_h *has an attracting cycle of period* 2.

Proof. We derive conditions on the coefficients of

$$p(z)=\sum_{k=0}^{n} a_k z^k$$

such that

$$N_h(0)=1,\ N_h(1)=0,\ N_h'(0)=0,\ N_h'(1)\neq\infty.$$

Then N_h has the (superattractive) periodic cycle $\{0, 1\}$. The four conditions are seen to be equivalent to

$$\begin{aligned}
&ha_0 = -a_1 \text{ and } a_1 \neq 0,\\
&-ha_0 + (1-h)a_1 + (2-h)a_2 + \cdots + (n-h)a_n = 0,\\
&(1-h)a_1^2 + 2ha_0a_2 = 0 \quad \text{and} \quad a_1 \neq 0,\\
&a_1 + 2a_2 + 3a_3 + \cdots + na_n \neq 0.
\end{aligned}$$

Clearly, these conditions can be satisfied for any $0 < h \leqslant 1$ if $n \geqslant 3$, e.g. choose

$$p(z) = (2-h)z^3 - (1-h)z^2 - 2z + \frac{2}{h}. \qquad \square$$

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