computer graphics in mathematics: viewing parametrized surfaces

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1. what can computer graphics do for mathematics?

Computers and in particular computer graphics have long become valuable tools for researchers in engineering and in the natural sciences. Molecular modeling, CAD/CAM and image processing are just some of the many disciplines in which computer graphics has been established as an intrinsic and indispensable element. But it is also offering significant potential to workers in the more abstract realms of mathematics, and in fact it already has provided insights that led to major discoveries most notably in dynamical systems and differential geometry. In 1964 the meteorologist E. N. Lorenz discovered the first chaotic attractor in an attempt to solve "the problem of deducing the climate from the governing equations" [Lor64]. Although he did not solve the problem, his computer graphical discovery of a strange attractor which is now named after him, inspired a whole generation of physicists, mathematicians and other scientists to study systems that previously had been discarded as being too complex and inaccessible.

Only in 1980 B. Mandelbrot at IBM and his coworkers ran some innocent computer experiments motivated by the 70 years old deep theory of the French mathematicians G. Julia and P. Fatou [Man80]. They studied families of iterated rational maps constructed in such a way that some of them exhibit chaos similar to the chaos found in the Lorenz attractor. What he found, was a somewhat fuzzy looking set with a few apparent "specks of dirt" around it which however failed to vanish in an attempt to clean up the picture by using better computer equipment. Little did he know that soon his name would be attached to this unusual set which now has emerged to become the "most complicated object in mathematics" as well as the favorite object in computer programs written by countless amateur and many professional scientists around the world. The "specks of dirt" as we now know, of course, were little satellite Mandelbrot sets connected by infinitely thin strands to the main body of the Mandelbrot set. For an interesting account of the history of the discovery of the Mandelbrot set, see Mandelbrot's contribution [Man86] in *The Beauty of Fractals* by H. O. Peitgen and P. Richter [Pei86].

Where the above two examples fall into the category of dynamical systems, the computer graphical discovery of a new so called minimal surface in 1985 has excited the world of differential geometry. At Rice University in Houston D. Hoffman and W. Meeks investigated a particular surface in three dimensional space of a certain topological type. It was already known to be minimal and complete, i.e. roughly speaking it is a surface of least area and without boundary. Computer graphics confirmed their conjecture, that it is also a surface without self-intersections, and in the following they were able to prove that fact, which makes that surface the third known example in its category besides the helicoid and the catenoid and the first new one to be found in over 200 years [Hof85, Pet85].

the only way to learn mathematics is to experience it

The above examples show that the real power of computer graphics comes in where it allows to visualize constructs that otherwise exist only in the pure form of mathematical abstractions. Another striking example is displayed on the cover of this year's January issue of *The Mathematical Intelligencer* : the hypersphere. The hypersphere is the straight forward generalization of the usual sphere in the four dimensional Euclidean space. Yet it is so hard to imagine, since our ways of thinking are accustomed to the only three dimensional space that surrounds us. However, taking certain cuts of the hypersphere and their stereographic projections facilitates computer graphical viewing and aids in understanding the geometry of the underlying object [Koc87]. Below we will return to the hypersphere discussing a particularly interesting surface that "lives" in it: the Klein bottle.

A computer graphics workstation such as the IRIS 3030 is an ideal tool for the experimental investigation of such mathematical objects. It may well provide the testing grounds for conjectures, and, when used wisely may suggest a wealth of new ideas and problems. A decade ago there were probably less than a handful of sophisticated graphics devices to be found in the mathematics departments of this country. Now the potential of computer graphics has transformed into a need, and new academic programs in computational mathematics including mathematical computer graphics are mushrooming at universities in many countries. But the benefits of computer graphics in mathematics are not restricted to research alone. Viewing of curves, surfaces, etc. in two or three dimensions and the interactive, realtime simulation of dynamical systems may provide the student with intuitive and lively insights in the theory that previously could have been communicated only via dry lectures and books. The author is convinced that in the not so far future traditional courses in analysis, differential equations, geometry, numerical analysis will be complemented and enriched by hands-on computer graphics projects. It has offen been stressed by teachers that "the only way to learn mathematics is to do mathematics". Now we are tempted to append "... and to experience it"!

2. viewing parametrized surfaces

It is out of the scope of this paper to discuss a computer graphical application from the frontiers of research and refer the reader to the literature listed below. More modestly we restrict to a simple yet interesting problem, namely the interactive display of parametrized surfaces, a task ideally suited for the IRIS workstation. Of course, parametrized surfaces are used widely in computer graphics, and we do not intend to contribute new techniques for the already initiated computer graphics user. In the following we will briefly describe the concept of parametrized surfaces and their interactive display. Moreover, we will demonstrate some of the diverse examples along with formulas such that the interested reader may well reproduce most of our pictures. This application has proven to be very useful as part of an introductory graphics course for mathematics students and as a demonstration tool in a differential geometry course at the University of California, Santa Cruz.

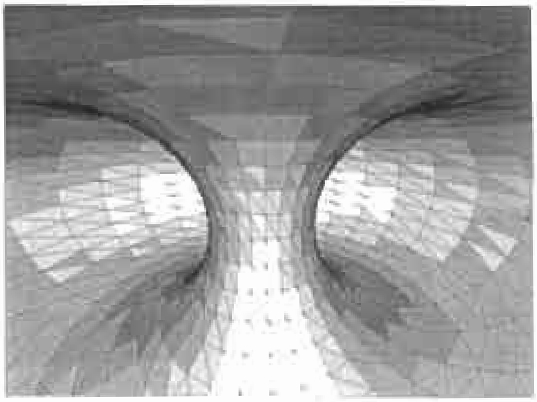
A first definition of a parametrized surface is given by a continuous map X with an (open) subset U of the plane R^2 as its domain and taking values in the 3-dimensional Euclidean space R^3. Typical examples are graphs of functions $f : R^2 \to R$ such as $f : (u,v) = \exp(-(u^2+v^2))$ yielding a bell shaped (infinite) surface. Here the map X would be written as $X(u,v) = (u, v, f(u,v))$. There are examples of nondifferentiable parametrized surfaces, such as fractal landscapes, however, usually it is required that X satisfies some differentiability conditions. If additionally the surface admits a tangent plane at all points $X(u,v)$, i.e. more precisely, if the derivative of X is nonsingular at all points $(u,v) \in U$, then the surface is called a regular, parametrized surface. For an introduction to differential geometry which includes the theory of regular surfaces we recommend [Car76]. The well known torus is a good example for a regular, parametrized surface. Here

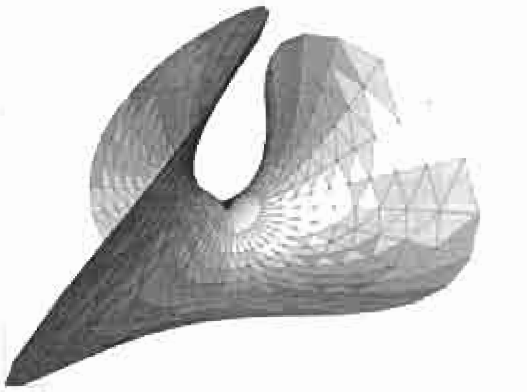
$$X(u,v) = ((r \cos u + R)\cos v, (r \cos u + R)\sin v, r \sin u),$$
$$0 < u < 2\pi, 0 < v < 2\pi.$$

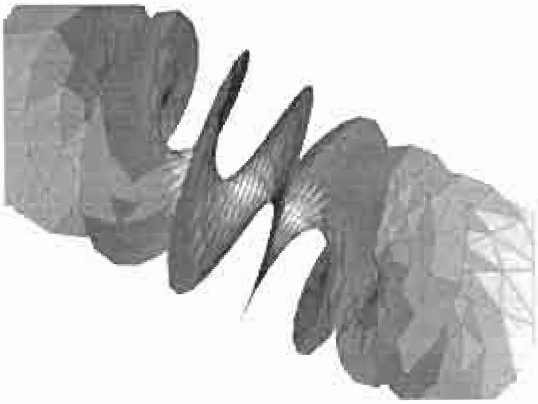
The surface is obtained by rotating a circle of radius r in the xz-plane and center $(R, 0, 0)$, $R > r$ around the z-axis. In this example the domain U of X is the square of length 2π. In a computer program which should be capable of displaying a whole palette of surfaces it should be convenient to parametrize all surfaces over the unit square $(0,1) \times (0,1)$. This is possible for all of the examples discussed here. For the torus we would simply scale up our (u,v)-coordinates of the unit square by a factor of 2π before applying X. The next step is to impose a rectangular grid on the unit square. After choosing the numbers n_u and n_v of subdivisions on the u- and v-axis we obtain $(n_u + 1)(n_v + 1)$ equally spaced grid points

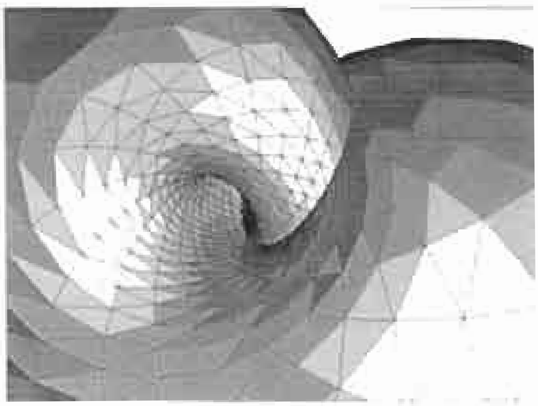
$$(u_i, v_j), i = 0, \ldots, n_u, j = 0, \ldots, n_v$$

from top to bottom:
View from inside the torus with radii 1 and 3
Enneper's surface parametrized by polar coordinates
The helicoid
Mouth of the Klein bottle, self intersection near center









with

$$u_i = i / n_u, v_j = j / n_v$$

in the domain U of X and corresponding points $X(u_i, v_j)$ on the parametrized surface. A wireframe model of the surface can now easily be drawn as a family of coordinate lines, effectively mapping the grid of the unit square onto the surface. The picture can be made clearer by choosing different colors of the coordinate lines in the u- and v-direction. Also, depth-cueing enhances the 3-dimensional character of the object.

For the purpose of shading we may further subdivide the grid squares $(u_i, u_{i+1}) \times (v_j, v_{j+1})$ into two triangles which are then mapped to the triangles defined by the vertices .

$$X(u_i, v_j), X(u_i+1, v_j), X(u_{i+1}, v_{j+1})$$

and

$$X(u_i, v_j), X(u_{i+1}, v_{j+1}), X(u_i, v_{j+1}).$$

We can now define consistent normal vectors for all triangles by computing the corresponding cross products, which in conjunction with a simple shading model yield values for flat polygon shading. For the smooth Gouraud shading surface normals for each of the vertices $X(u_i, v_j)$ have to be calculated. These may either be computed as normalized cross products of the partial derivatives $X_v(u_i, v_j)$ and $X_u(u_i, v_j)$, if available or via numerical approximation of these derivatives.

The hidden surface problem is most easily resolved by the Z-buffering technique supported by the IRIS workstation. However, the simplest version of the painter's algorithm is often sufficient as our pictures here prove. In that algorithm we compute the barycenters of all triangles, then sort them according to the distance from the eye position for perspective viewing or from a plane orthogonal to the parallel projection vector. Finally the sorted polygons are rendered in the order of decreasing distance. This algorithm actually has two advantages over the more precise Z-buffering technique. It is generally faster if the number of polygons is not larger than a couple of thousands. Moreover, it is very instructive to watch the computer draw the polygons from back to front. In this way the geometry of more complicated surfaces like the Klein bottle become much clearer. To profit from advantages of both methods, of course, we can render the sorted polygons from back to front with Z-buffering in effect as well.

3. examples of parametrized surfaces

In the following we list formulas for the generation of most of the pictures accompanying this article. The first example, the torus, is already given in the previous section.

minimal surfaces

The helicoid and Enneper's surface are two particularly accessible examples of minimal surfaces, i.e. surfaces with vanishing mean curvature. The parametrization of the helicoid is

$$X(u, v) = (v \cos u, v \sin u, u)$$

and Enneper's surface is given by

$$X(u, v) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2)$$

The coordinate lines in our figure are the polar coordinate lines for $u = r \cos \phi$, $v = r \sin \phi$. In Enneper's surface we have $r \leq 1.5$ and for larger values of r we obtain self-intersections in the xz- and the yz-plane.

the Moebius strip

The Moebius strip is obtained by taking a rectangular planar strip, twisting its ends by half a rotation and gluing them together. Its relevance stems from the fact, that it is the prototype example for a non-orientable surface, i.e. one cannot define a differentiable field of surface unit normal vectors on the whole surface. A system of coordinates for the Moebius strip is given by

$$X(u, v) = ((2 \cdot v \sin \frac{u}{2}) \sin u, (2 - v \sin \frac{u}{2}) \cos u, v \cos \frac{u}{2})$$

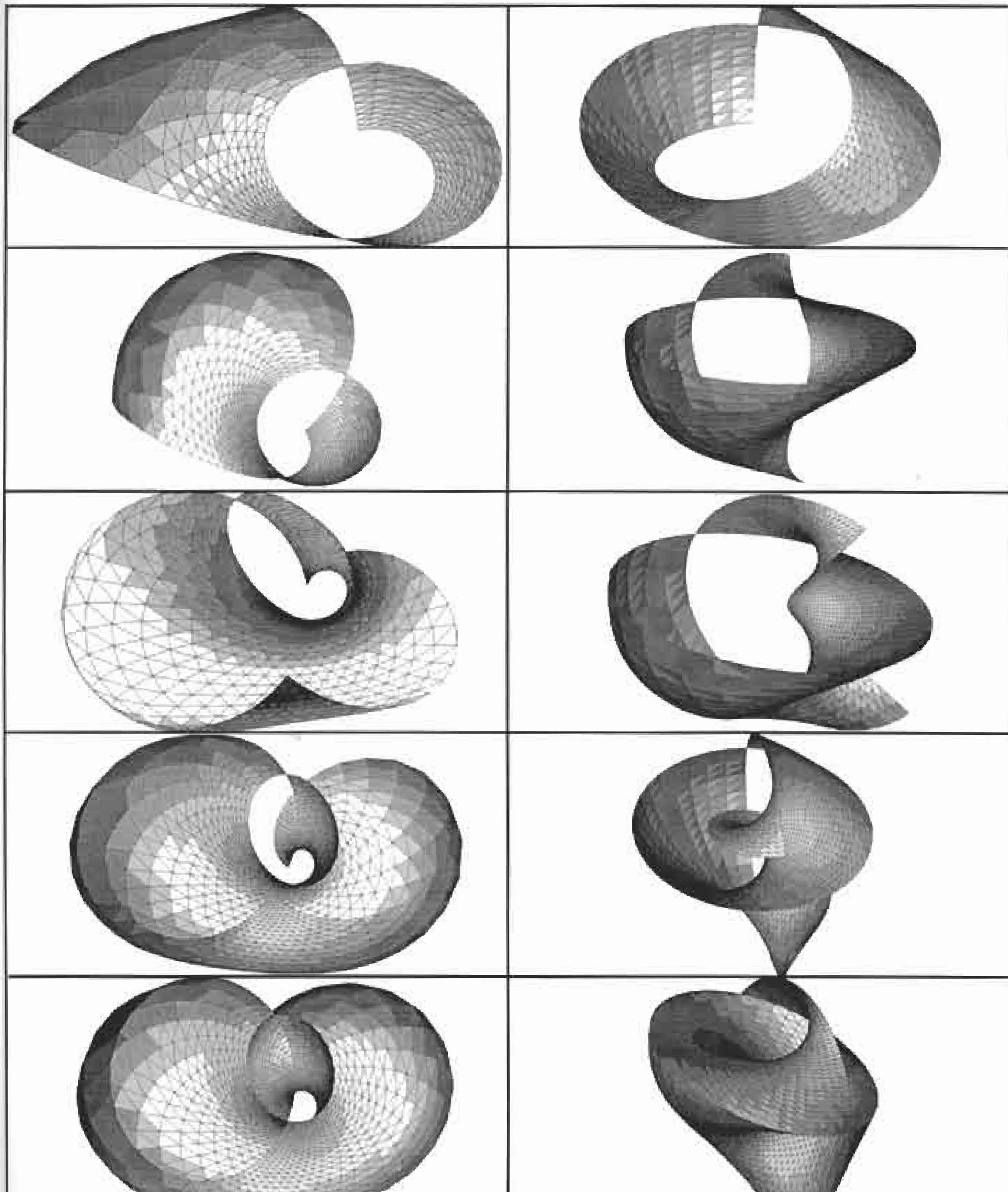
with $0 < u < 2\pi$ and $-1 < v < 1$.

figures Klein 1a, 1b, ..., 5a, 5b

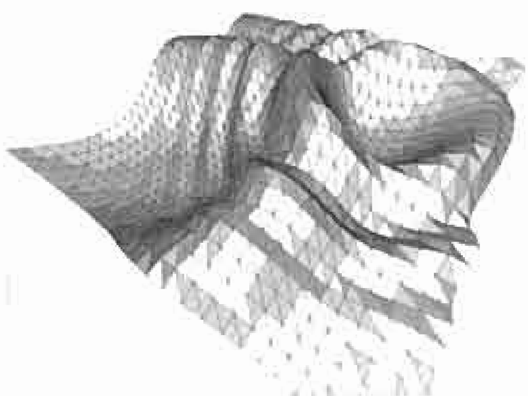
In the oppositetwo series of figures we have split the Klein bottle K into two parts : $K(u,v)$ with $v \leq \pi$ and $K(u,v)$ with $v \geq \pi$. Let us denote by $K_{i,j}$ the part of the Klein bottle with coordinates $0 \leq u \leq 2\pi$ and $2\pi i/10 \leq v \leq 2\pi j/10$. Then the complete bottle is the union of $K_{0,5}$ and $K_{5,10}$. These sets are shown in figures 5a and 5b. $K_{i,j}$ is one of the Moebius strips in the Klein bottle, which we also cut into two halfs, namely $K_{0,1}$ (fig. 1a) and $K_{9,10}$ $(=K_{-1,0})$ (fig. 1b). The above sequences show how one obtains the two halfs of the figures 5a,b from the two half Moebius strips of the figures 1a,b by consecutively adding parts of the surface : $K_{0,2}$ (2a), $K_{0,3}$ (3a), $K_{0,4}$ (4a), $K_{0,5}$ (5a) and $K_{8,10}$ (2b), $K_{7,10}$ (3b), $K_{6,10}$ (4b), $K_{5,10}$ (5b). Note that a full circle, namely the coordinate line $v = 0$ (same as $v = 2\pi$) is part of all the surfaces shown.

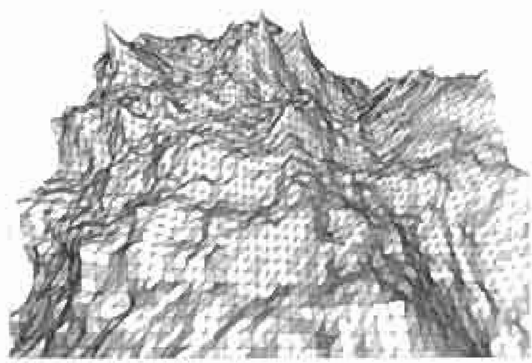
Try to follow the evolution of the boundary of the surface from figures 1a,b to 5a,b. In 1a,b the boundary is composed of the circle and three sides, and we have three vertices, one on the circle (reached twice) and two other vertices (corners). As we progress down to 5a,b these two corners both tend to the antipode of the first vertex on the circle. In the finishing stage we have that the boundary consists of the circle, which is traversed twice accounting for the self-intersection of the Klein bottle and another larger circle which meets the first one in the two antipodal vertices. To assemble the complete Klein bottle, we have to mate the two doubly traversed circles of 5a,b (due to different scalings the one in 5b seems to be larger than the one in 5a) and also the two other circles. Of course it is best to actually manipulate wireframes of both halfs to appreciate the full complexity and beauty of this classical surface.

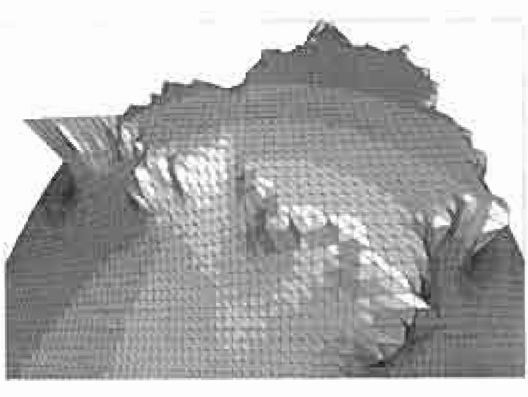
Klein bottle figures
Left row: Figures 1a-5a   *Right row: Figures 1b-5b*



from top to bottom.
Moebius strip
Summation of two "cowboy hats"
Fractal surface generated by midpoint displacement
Potential of the full Mandelbrot set







the Klein bottle

The Klein bottle is an abstract regular surface obtained from the usual torus in 3-space by identifying a point (x, y, z) on the torus with its antipode $(-x, -y, -z)$. One can represent this abstract surface as regular parametrized surfaces in 4-space. In order to project the surface into 3-space it is convenient to use the following 4-dimensional version:

$$K(u, v) = (\cos u \cos v \,,\ \sin u \cos v \,,\ \cos\frac{u}{2} \sin v \,,\ \sin\frac{u}{2} \sin v)$$

where u and v range from 0 to 2π. Here the identification of antipodal points is established by

$$K(u, 0) = K(u, 2\pi) \text{ and } K(0, v) = K(2\pi, 2\pi - v).$$

This implies that the image of the strip

$$0 \leq u \leq 2\pi \text{ and } \pi - \varepsilon < v < \pi + \varepsilon$$

under K is a Moebius strip, and therefore the Klein bottle cannot be oriented. Actually there is another Moebius strip given by the parameters

$$0 \leq u \leq 2\pi \text{ and } -\varepsilon < v < \varepsilon.$$

Moreover, since

$$K(u, 0) = K(u + \pi, \pi) = K(u, 2\pi)$$

we have that the coordinate lines $v = 0$, $v = \pi$ and $v = 2\pi$ are all the same, namely the unit circle in the xy-plane and this is also the common center line of the two Moebius strips. Therefore it seems to be natural to to cut away the two Moebius strips as in some of our pictures in order to get a better understanding of the 3-dimensional geometry of the Klein bottle.

We have seen that our particular parametrization K does not yield an embedding in 4-space, since self-intersections occur. However, we have the advantage, that $K(u, v)$ is always a point in the hypersphere S^3, i.e. if (x, y, z, w) is a point on the surface, then $x^2 + y^2 + z^2 + w^2 = 1$. Therefore we can use stereographic projection P to identify a point (x, y, z, w) in S^3 with
This will work for all points in S^3 except for the north pole $(0, 0, 0, 1)$

$$P(x, y, z, w) = (\frac{x}{1-w}, \frac{y}{1-w}, \frac{z}{1-w}) \in R^3$$

of S^3 which would be mapped to ∞. Unfortunately it is the case that the north pole is a point of the Klein bottle (set $u = \pi$, $v = \pi/2$), and, thus, stereographic projection would lead to an unbounded surface in 3-space. As a solution to this problem we can rotate the Klein bottle in the hypersphere S^3 such that a new point that is not in the surface is rotated into the north pole. E.g. $(1/\sqrt{3}, 0, 1/\sqrt{3}, 1/\sqrt{3})$ is a good choice for such a point. If T denotes this rotation, then the final unit square parametrization of the Klein bottle in 3-space is

$$X(u, v) = P(T(K(2\pi u, 2\pi v))).$$

cowboy hats

As a modified version of the popular "cowboy hat" we have used the graph of the function

$$f(u, v) = \sum_{i=1}^{N} a_i e^{d_i r_i^2} \cos(f_i \, r_i^2)$$

with

$$r_i^{\,2} = (u - u_i)^2 + (v - v_i)^2$$

where

N = number of "hats" to be displayed,
(u_i, v_i) = center of the i-th "hat",
r_i = squared distance of (u, v) from center (u_i, v_i),
d_i = damping factor (> 0),
f_i = frequency factor,
a_i = amplitude.

the potential of the Mandelbrot set

The Mandelbrot set is a prototype model for the transition from order into chaos. The set in its usual definition is embedded in the complex plane. The key idea is that each point $c \in \mathbf{C}$ of the plane determines a dynamical system R_c whose characteristic properties can be derived by checking whether c is in the Mandelbrot set or not. In this line of thought the Mandelbrot set M may be viewed as a complicated road map of dynamical systems (see [Pei86]). Its formal definition is

$$M = \{ c \in \mathbf{C} \mid \lim_{k \to \infty} R_c^k(c) \neq \infty \}$$

where

$$R_c(z) = z^2 + c$$

$$R_c^k(z) = R_c(R_c(\ldots R_c(z))), \ (k\text{-times}).$$

If the limit in the above definition does not exist, then we also take $c \in M$.

If one supposed the Mandelbrot set being metallic and charged with electricity, then it would induce an electrostatic field in its neighborhood. A probe introduced near the Mandelbrot set then would be subject to a certain force due to the potential. The lines of points exhibiting an equal amount of this attracting force are the equipotential lines, and these are basically the lines of equal color seen in the many color pictures of the Mandelbrot set.

In the standard algorithm for the computation of a picture an integer $k(c)$ which later will serve as an index for a color look up table is computed for a point $c \in \mathbf{C}$ via

```
k = 0;
while ( |R_c^k (c)| < K and k < k_max )
      k = k + 1;
```

Here k_{max} is the maximal number of iterations we allow for each point (pixel) and K should be a large number such as 1000 (try also $K = 2.25$ and note the difference). The function $k(c)$ is only piecewise constant and does not yet yield a smooth parametrized surface. To achieve a smooth equivalent of $k(c)$ we may define the potential function $H(c)$ as

$$H(c) = \lim_{k \to \infty} 2^{-k} \log |R_c^k(c)|.$$

This limit converges very rapidly once $|R_c^k(c)|$ is large. Thus it suffices to iterate only until $k = k(c)$. For points c very close to the Mandelbrot set we need many iterations ($k(c)$ is large) and therefore the corresponding potential is $H(c)$ is very small. For points $c \in M$ we have $H(c) = 0$. The parametrized surface defined as the graph of H then is given by $X(u, v) = (u, v, H(u+iv))$. Similar potentials can be defined for the basin of attraction of ∞ for a fixed parameter $c \in \mathbf{C}$ in $R(z) = z^2 + c$, see [Pei86] and our figures.

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