Global Bifurcation of Periodic Solutions to Some Autonomous Differential Delay Equations

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ABSTRACT

We are concerned with slowly oscillating, periodic solutions of the parametrized differential delay equation $\dot{x}(t) = -\lambda f(x(t-1))$, $\lambda > 0$, where f is an odd, continuous, and nonlinear function satisfying $xf(x) > 0$ for all $x \neq 0$. Due to the oddness of f we can identify odd-harmonic periodic solutions having the period T with other solutions of the periods $2T/(T-2)$. From this observation and a result of Nussbaum we obtain the existence of periodic solutions with a period close to 2. Moreover, we can reach conclusions about a secondary bifurcation of periodic solutions. We present a combination of a Galerkin scheme with a continuation method which yields a feasable numerical procedure for the effective computation and continuation of periodic solutions. We report the results of an extensive case study for the nonlinearities $f(x) = x/(1+|x|^p)$, $p \geqslant 1$. For $p = 8$ we show many different continua of periodic solutions which are either disjoint or bifurcating from continua of periodic solutions that contain a higher symmetry. For $p = 1$ and $\lambda > \pi/2$ it is known that periodic solutions are unique. We discuss two homotopies linking the cases $p = 8$ and $p = 1$ which exhibit very different transitions from multiplicity to uniqueness of solutions.

1. INTRODUCTION

In the past two decades a lot of research has been done on the structure of continua of periodic solutions of delay equations e.g. of the type $\dot{x}(t) = f(x(t), x(t-\tau))$. Such equations are often motivated directly from the natural sciences: In 1977 Mackey and Glass proposed the differential delay equation

$$\dot{x}(t) = \frac{ax(t-\tau)}{1+[x(t-\tau)]^8} - bx(t), \qquad a, b, \tau > 0, \tag{1}$$

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as a model for the dynamics of the production of red blood cells. The growth rate in (1) is assumed to depend on the concentration of cells at time $t - \tau$. If the delay time τ is sufficiently large, as it is conjectured to be in patients with leukemia, the concentration $x(t)$ will oscillate or even behave chaotically ([9]). Yorke (see [16]) first considered the simplified model

$$\dot{x}(t) = -\lambda \frac{x(t-1)}{1+|x(t-1)|^p}, \qquad p \geqslant 1, \quad \lambda > 0, \tag{2}$$

which seems to generate very similar behavior of solutions for $p = 8$ and sufficiently large parameters λ.

We will give a comprehensive numerical case study of Equation (2) which, even though simple looking, already admits many different periodic solutions and complex dynamical behavior. The main motivation for favoring Equation (2) over (1) is that very little rigorous has been done for Equation (1), whereas there are some results for (2). In this paper we are following and extending the discussion in [20].

Let us first briefly describe the main theoretical results for the more general equation

$$\dot{x}(t) = -\lambda f(x(t-1)), \qquad \lambda > 0, \tag{3}$$

with a continuous nonlinearity $f: R \to R$ satisfying

$$\begin{aligned} f(-x) &= -f(x) && \text{for all} \quad x \in \mathbb{R}, \\ f(x) &> 0 && \text{for all} \quad x > 0. \end{aligned} \tag{H}$$

Let $\varphi: [-1,0] \to \mathbb{R}$ denote a continuous initial function for the infinite-dimensional initial-value problem

$$\begin{aligned} x(t) &= -\lambda f(x(t-1)) && \text{for} \quad t \geqslant 0, \\ x(t) &\equiv \varphi(t) && \text{for} \quad -1 \leqslant t \leqslant 0. \end{aligned} \tag{4}$$

There is a unique solution x_φ of (4) defined on the interval $[-1, \infty)$ which one may readily obtain via consecutive integration over unit intervals.

We call a solution x of (3) or x_φ of (4) *slowly oscillating* if it has infinitely many zeros and if the distance between any two zeros is always greater than 1. It is not hard to see that if f is differentiable at 0 and $f'(0) > 0$, then we have for $\lambda > f'(0)^{-1}$ that any initial function $\varphi \neq 0$ from the set

$$P = \{ \varphi \in C[-1,0] | \varphi(-1) = 0, \ \varphi \text{ monotonically increasing} \}$$

generates a slowly oscillating solution x_φ of (4) [11].

For the study of slowly oscillating periodic solutions one may use the following Poincaré-type operator (see Figure 1).

DEFINITION 1. For $\varphi \in P$ let x_φ denote the corresponding solution of (4) and z_1 the first nonnegative zero of x_φ. The *shift operator* $\mathcal{S}_\lambda : P \to P$ is defined by

$$\mathcal{S}_\lambda(\varphi) : t \to \begin{cases} -x_\varphi(z_1 + 1 + t) & \text{if } z_1 \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

$\mathcal{S}_\lambda$ is a continuous compact operator [11], but it is not Frèchet differentiable at $0 \in P$. Since f is an odd function, we have that a nontrivial fixed point of the shift operator $\mathcal{S}_\lambda$ or of one of its iterates $\mathcal{S}_\lambda^k$, $k = 2,3,4,\ldots$, induces a slowly oscillating periodic solution of (3), which we call an *S-solution* or *S^k-solution* respectively. It is clear that an S-solution is also an S^k-solution for

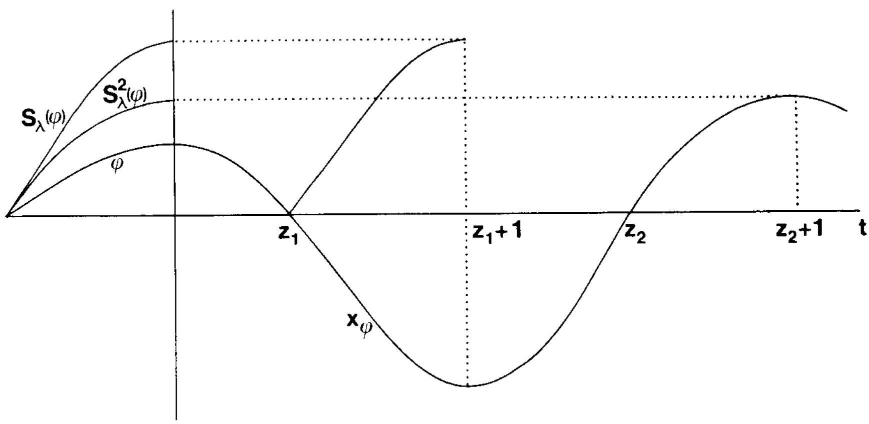

FIG. 1. The shift operator $\mathcal{S}_\lambda$ and its iterate $\mathcal{S}_\lambda^2$.

all positive integers k, whereas there may be S^k-solutions which are not S^j-solutions for any $j < k$.

Due to the oddness of f, there exist *special S-solutions* which are characterized by the following symmetry property. If t_0 denotes a zero of the solution x of (3), then we have

$$x(t_0 + t) = x(t_0 + 2 - t)$$

for all $t \in R$. Thus, special S-solutions are sinusoidal and have the period 4. The following result about the global existence of special S-solutions holds.

THEOREM 2 [1]. *Let f be as in* (H). *The set of initial functions which yield special S-solutions of* (3)

$$\Phi = \{ (\varphi, \lambda) \in P \times \mathbb{R}^+ | \varphi \neq 0, \mathbb{S}_\lambda(\varphi) = \varphi, x_\varphi(t) = x_\varphi(-t) \text{ for } |t| \leqslant 1 \}$$

is a curve in $P \times \mathbb{R}^+$ which can be parametrized above $\|\varphi\|$ $(= \varphi(0))$. If f is differentiable at 0 *with $f'(0) > 0$, then we have that $(\varphi, \lambda) \in \Phi$ and $\|\varphi\| \to 0$ implies*

$$\lambda \to \frac{\pi}{2} f'(0)^{-1}.$$

Thus, if we regard the trivial solution $x \equiv 0$ of (3) as a periodic solution, we can think of $(0, (\pi/2)f'(0)^{-1})$ as a (Hopf) bifurcation point for periodic solutions of (3). Here a continuum of special S-solutions emanates.

Results from [7] in connection with [14] yield conditions on the nonlinearity f which guarantee that the above special S-solutions are the only slowly oscillating periodic solutions of (3):

THEOREM 3 [14, 15]. *In addition to* (H) *assume that*

(*i*) *f is bounded,*

(*ii*) *f is a C^1 function such that*

$f'(x) > 0$ *for all* $x \in R$,

$f'(1) = 1$,

$f'(x)$ *is monotonically decreasing for* $x \geqslant 0$,

(*iii*) $f(x)/x$ *is strictly monotonically decreasing for* $x > 0$.

Then for $0 < \lambda \leqslant \pi/2$ Equation (3) *has no slowly oscillating periodic solution, and for $\lambda > \pi/2$ there exists exactly one slowly oscillating periodic solution.*

This solution (a special S-solution) is globally attractive, i.e., for a given $\varphi \in P - \{0\}$ the sequence $\mathcal{S}_\lambda^k(\varphi)$, $k = 1, 2, \ldots$ converges to a fixed point $\varphi^* = \mathcal{S}_\lambda(\varphi^*)$ which corresponds to the above periodic solution.

It is clear that the nonlinearity f from Equation (2) satisfies all the conditions of Theorem 3 if we set $p = 1$. Thus, in this case we have uniqueness of slowly oscillating periodic solutions. However, for $p > 1$ Theorem 3 does not guarantee uniqueness, and in fact for $p > 3$ there must exist other periodic solutions:

THEOREM 4 [16]. *Let f be as in* (H). *Let $x_1 > 0$, and let f be monotonically increasing on $[0, x_1]$ and monotonically decreasing on $[x_1, \infty)$. Let b, c, d, r, x_0 be positive constants such that*

$$(c - dx^{-b})x^{-r} \leqslant f(x) \leqslant (c + dx^{-b})x^{-r}$$

holds for all $x \geqslant x_0$. If $r > 2$ and $b > r/(r-1)$, then there exists $\lambda_0 > 0$ such that for all $\lambda \geqslant \lambda_0$ there exists an S-solution y_λ of (3). *The (minimal) period $T(\lambda)$ of y_λ tends to ∞ as $\lambda \to \infty$.*

One checks that all the above hypotheses are satisfied if we choose $f(x) = x/(1 + |x|^p)$ as in Equation (2) and $p > 3$.

Let the period always mean the minimal period of a function. In Section 2 we will construct a transformation which for a given S-solution with period T will generate another S-solution with period $\tilde{T} = 2T/(T-2)$. Applying this observation to the S-solutions of Theorem 4, we immediately obtain the existence of S-solutions whose periods tend to 2 as $\lambda \to \infty$. Another consequence deals with a possible and numerically verified secondary bifurcation of S-solutions from the continuum of special S-solutions: Under certain assumptions the bifurcation has to be a backward bifurcation.

In Section 3 we will present an efficient numerical procedure for the approximation and continuation of periodic solutions, which we will use in Section 4 for the numerical study of e.g. the following questions:

1. What is the complete bifurcation diagram for S-solutions of (2)? Are the S-solutions of Theorem 4 linked with other S-solutions?
2. How does the bifurcation diagram change if we let $\varepsilon > 0$ and $f(x) \to \pm\varepsilon$ as $x \to \pm\infty$?

3. In particular, how does the transition from multiplicity of periodic solutions for $p = 8$ to uniqueness at $p = 1$ occur? Does the transition depend on the homotopy that one chooses to connect the two cases?

4. What is the bifurcation structure of S^2- an S^4-solutions of (2) and their dependence on p?

Nussbaum has conjectured a sequence of cascading bifurcations of periodic solutions of (2) in the following sense:

CONJECTURE 5 [12]. There is a sequence $(\lambda_k)_{k=0,1,2,\dots}$ with $\lambda_0 = \pi/2$ and $\lambda_i < \lambda_j$ for $i < j$ such that S-solutions of (2), $p = 8$ bifurcate at λ_0, S^2-solutions bifurcate from S-solutions at λ_1, S^4-solutions bifurcate from S^2-solutions at λ_2, etc (see Figure 2).

Using the continuation method, we confirm this conjecture for S-, S^2-, and S^4-solutions. In this cascading bifurcation the periodic solution with the largest period is attractive with respect to the integration of the differential delay equation. This enables us to check the conjecture up to S^{64}-solutions.

Also, by numerical integration one finds parameter ranges of λ where other stable and attractive periodic solutions (e.g. S^3-solutions) dominate, and other ranges with apparently chaotic behavior. A more complete study of these aspects of Equation (2) has been started in [20] and will appear elsewhere.

Let us remark that recently Peters [18] has modeled the nonlinearity in (2) by a simple piecewise constant function. This approach reduces the infinite-dimensional equation (2) to a finite-dimensional one and allows a complete

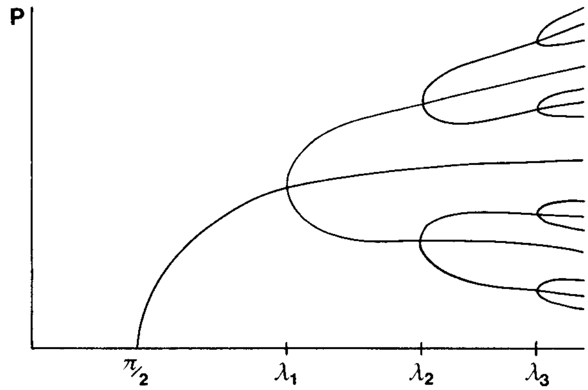

FIG. 2. Cascading bifurcation of fixed points of $\mathcal{S}_\lambda^{2^k}$: At $\lambda = \lambda_k$ fixed points of $\mathcal{S}_\lambda^{2^k}$ bifurcate from fixed points of $\mathcal{S}_\lambda^{2^{k-1}}$.

analysis of the fixed points and dynamics of the corresponding shift operator, which shows complicated and chaotic behavior.

2. ODD NONLINEARITIES IMPLY CONJUGATE S-SOLUTIONS

It is an observation due to Cooke (see [5]) that a given periodic solution of (3) with period T may be transformed to other periodic solutions with smaller periods $T/(4k+1)$, $k = 1, 2, \ldots$. This can be achieved by means of a linear scaling in time and a modification of the parameter λ. In the case of an odd nonlinearity f we obtain another kind of transformation of odd-harmonic periodic solutions by applying the same technique and additionally reversing the orientation of time. For completeness we summarize both transformations.

PROPOSITION 6. *Let $f\colon R \to R$ be a continuous function and $x\colon R \to R$ a periodic solution of $\dot{x}(t) = -\lambda f(x(t-1))$ of period T. Assume either* (i) *or* (ii):

(i) (*Cooke*) *Choose $k \in \{1, 2, \ldots\}$ and set*

$$a = kT + 1,$$

$$\tilde{x}\colon R \to R, \qquad \tilde{x}(t) = x(at).$$

(ii) *Assume that f is an odd function and that x is an odd-harmonic solution, i.e. $x(t + T/2) = -x(t)$ for all $t \in R$. Further assume $T > 2$ and set*

$$a = \frac{T-2}{2},$$

$$\tilde{x}\colon R \to R, \qquad \tilde{x}(t) = -x(-at).$$

Then $\tilde{x}$ is a periodic solution of $\dot{\tilde{x}}(t) = -\tilde{\lambda} f(\tilde{x}(t-1))$ for $\tilde{\lambda} = a\lambda$. The period of $\tilde{x}$ is $\tilde{T} = T/a$.

PROOF. Trivial.

Assume for the remainder of this section that f satisfies (H). The hypotheses of Proposition 6(ii) are fulfilled in particular for S-solutions of (3). It is an immediate consequence that the transformation defines a bijective map Ψ on the set

$$D = \{(x, \lambda) \in C(R) \times \mathbb{R}^+ | 0 \neq x \text{ is an S-solution of } \dot{x}(t) = -\lambda f(x(t-1)) \text{ with } x(0) = 0\}$$

by letting $\Psi(x, \lambda) = (\tilde{x}, \tilde{\lambda})$. Ψ is bijective, because we easily check $\Psi \circ \Psi = \mathrm{Id}|_D$; thus $\Psi^{-1} = \Psi$. Moreover, it follows that $(x, \lambda) \in D$ is invariant under Ψ, i.e. $\Psi(x, \lambda) = (x, \lambda)$, if and only if x is a special S-solution. For these reasons we call $\tilde{x}$ the *conjugate S-solution* of x. Figure 3 demonstrates an S-solution of (2) with a high period and its conjugate S-solution.

We can apply Proposition 6(ii) also to other periodic solutions of (3), e.g. to S^k-solutions where $k \geqslant 3$ is an odd integer. However, in this case we do not obtain any slowly oscillating solutions.

For the S-solutions of Theorem 4 we have $T \to \infty$ for $\lambda \to \infty$. Therefore we conclude for the corresponding conjugate S-solutions that $\tilde{T} = 2T/(T-2) \to 2$. Thus, we obtain the existence of S-solutions with period arbitrarily close to 2.

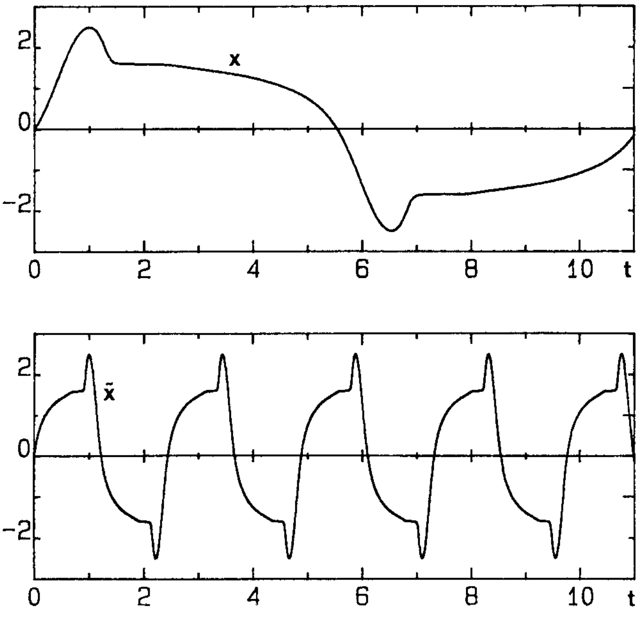

FIG. 3. S-solution x and conjugate S-solution $\tilde{x}$ of Yorke's equation (2) for $p = 8$. $\lambda = 5$ for the above plot and $\tilde{\lambda} \approx 22.5$ for the conjugate solution below.

Corollary 7. *Assume all hypotheses of Theorem* 4. *For* $\varepsilon > 0$ *there exists an S-solution of period* T *with* $2 < T < 2 + \varepsilon$.

We remark that it is also proven in [16] that under additional, technical conditions on f and in the case $0 < r < 2$, the S-solutions of large periods no longer exist. More precisely, the periods of S-solutions are bounded from above by a constant $T_{\max}$ depending only on the nonlinearity f. If we define $T_{\min} = 2T_{\max}/(T_{\max} - 2)$, it is clear that the periods of S-solutions are bounded from below by $T_{\min} > 2$. To see this, just observe that an S-solution with a period less than $T_{\min}$ would have a conjugate S-solution with a period exceeding the upper bound $T_{\max}$. By the same argument we obtain:

Corollary 8. *Let* f *be as in* (H) *and let* $T > 2$. *There is an S-solution of* (3) *with period* T *if and only if there is an S-solution with period* $\tilde{T} = 2T/(T-2)$.

Thus, e.g., if there are no S-solutions with period 6, there cannot be any S-solutions with period 3.

Our last consequence of the existence of conjugate S-solutions concerns possible bifurcation points in the set of S-solutions. Let us make the following assumption:

(A) Let $(x_0, \lambda_0) \in D$ be a special S-solution, and let U be a neighborhood of (x_0, λ_0) in $C(R) \times R^+$. Assume that $D \cap U$ is given by two curves intersecting only at the bifurcation point (x_0, λ_0).

Of course, one of the two curves consists of special S-solutions. The other curve we call W, and it is given by (x_0, λ_0) along with two bifurcating branches of nonspecial S-solutions. If these branches define a tangent at (x_0, λ_0) in the corresponding bifurcation diagram, we will argue that the bifurcation must be a backward bifurcation.

Recall that $\Psi(x_0, \lambda_0) = (x_0, \lambda_0)$, and observe that the two branches of W are mapped onto each other via Ψ, because otherwise there would be more than two curves of S-solutions intersecting at the bifurcation point. Thus, we may assume a parametrization $w: (-\varepsilon, \varepsilon) \to C(R) \times R$, $\varepsilon > 0$, of W in a sufficiently small neighborhood of (x_0, λ_0), such that the relation $w(-s) = \Psi(w(s))$ holds for $|s| < \varepsilon$. For a given S-solution $w(s)$ let $\lambda(s)$ denote the value of the parameter λ and $T(s)$ the period of the solution.

LEMMA 9. *Assume* (A). *Let* f *satisfy* (H) *and a Lipschitz condition. For* $0 < |s| < \varepsilon$ *define*

$$g(s) = \frac{\lambda(s) - \lambda_0}{T(s) - 4}.$$

If $g(0) = \lim_{s \to 0} g(s)$ *exists, then* $g(0) = -\frac{1}{4}\lambda_0$.

PROOF. First note that an S-solution z has period 4 if and only if it is a special S-solution. To see this, assume $z(0) = 0$ and observe that the pair $(z(t), z(t-1))$ and its conjugate $(\tilde{z}(t), \tilde{z}(t-1))$ both solve the initial-value problem

$$\dot{x}(t) = -\lambda f(y(t)), \qquad x(0) = z(0),$$

$$\dot{y}(t) = \lambda f(x(t)), \qquad y(0) = z(-1).$$

With the Lipschitz condition we conclude the uniqueness of solutions of the above initial-value problem. Thus, $z = \tilde{z}$. Following the remark after Proposition 6, this equality implies that z must be a special S-solution. Therefore we have for $0 < |s| < \varepsilon$ that the period $T(s)$ of the S-solution given by $w(s)$ is not equal to 4 and thus g is defined for all such s. From Proposition 6 we conclude for $|s| < \varepsilon$

$$\lambda(-s) = \tfrac{1}{2}[T(s) - 2]\lambda(s),$$

$$T(-s) = \frac{2T(s)}{T(s) - 2}.$$

With these equalities we obtain for $0 < |s| < \varepsilon$

$$g(s) + g(-s) = \tfrac{1}{2}\lambda_0 - \tfrac{1}{4}\lambda(s)T(s).$$

To complete the proof we let $s \to 0$ and note that this implies $\lambda(s) \to \lambda_0$ and $T(s) \to 4$.

The geometrical interpretation of Lemma 9 is given in Figure 4. There the two curves of S-solutions are graphed in the (λ, T) plane. By definition of g

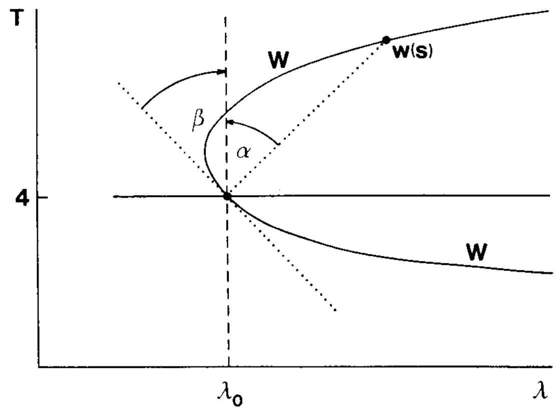

FIG. 4. Schematic diagram of backward bifurcation of S-solutions. T denotes the period of the solutions. The branch of S-solutions with $T \equiv 4$ contains only special S-solutions. We have $\tan\alpha = g(s)$ and $\tan\beta = -\frac{1}{4}\lambda_0$.

the angle α is given via $\tan\alpha = g(s)$, whereas the lemma states, that

$$\tan\beta = -\tfrac{1}{4}\lambda_0.$$

In particular, we have $\tan\beta < 0$, which means that the bifurcation is a backward bifurcation.

Let us additionally assume that the nonlinearity f is bounded and that the branch of S-solutions with period larger than 4 is unbounded in $C(\mathbb{R})\times\mathbb{R}$. Then this branch contains a turning point (x^*, λ^*), i.e., for all S-solutions (x, λ) of the branch we have $\lambda \geqslant \lambda^*$. This is true because for such (x, λ) we have $\|x\| \leqslant \lambda\|f\|$; hence the branch must contain S-solutions for arbitrary large parameters λ. In connection with the backward bifurcation at (x_0, λ_0) this yields the turning point.

In Section 4 we will present numerical evidence that the assumption (A) and its conclusions in fact do apply e.g. to Equation (2).

3. APPROXIMATION OF PERIODIC SOLUTIONS OF DELAY EQUATIONS

There seems to be very little literature on the numerical approximation of periodic solutions in delay equations. In [3, 6, 17, 20] a straightforward discretization of the shift operator is employed, and in [4] a related translation operator, which also requires the solution of the initial-value problem (4), is used. Completely different approaches are the various projection methods given in [2,20]. In [2] trigonometric polynomials and collocation on a set of implicitly defined mesh points are used. In the following we will briefly describe one of the Galerkin methods contained in [20] and some of its properties concerning conjugated solutions and bifurcation.

For simplicity, and since our particular interest is the numerical study of periodic solutions of (2), we restrict ourselves to the case (3) with f satisfying (H). However, it will be clear how the method can be generalized to suit other nonlinearities, the presence of several delays, or even systems of such equations.

Observe that a T-periodic solution of $\dot{x}(t) = -\lambda f(x(t-1))$ is equivalent to a 2π-periodic solution of

$$\dot{x}(t) = -\frac{\lambda}{\omega} f(x(t-\omega)), \qquad \omega = \frac{2\pi}{T}. \tag{5}$$

For notational convenience we will use (5) instead of (3). We call a 2π-periodic solution of (5) an S^k-solution if the above change of variables yields an S^k-solution of (3).

Let $C_{2\pi}$ be the function space

$$C_{2\pi} = \{ x \in C(\mathbb{R}) | x(t+2\pi) = x(t) \text{ for all } x \in R \},$$

and let $E_m \subset C_{2\pi}$ denote the $(2m+1)$-dimensional subspace of $C_{2\pi}$ which is given by all functions x_m of the form

$$x_m(t) = \tfrac{1}{2}a_0 + \sum_{k=1}^{m} a_k \cos kt + b_k \sin kt$$

with real coefficients $a_0, a_1, \ldots, a_m$ and $b_1, \ldots, b_m$. Introduce the operators

$$\mathcal{T}_\omega : C_{2\pi} \to C_{2\pi}, \qquad \mathcal{T}_\omega x(t) = x(t-\omega), \quad \omega \in \mathbb{R},$$

$$\mathcal{F} : C_{2\pi} \to C_{2\pi}, \qquad \mathcal{F}x(t) = f(x(t)),$$

and the projection

$$\mathcal{P}_m : C_{2\pi} \to C_{2\pi}$$

with $\mathcal{P}_m(C_{2\pi}) = E_m$, where the (Fourier) coefficients of $\mathcal{P}_m x$ are as usual given by

$$a_k = \frac{1}{\pi}\int_0^{2\pi} x(t)\cos kt\,dt, \qquad k = 0,1,2,\dots,m,$$
$$b_k = \frac{1}{\pi}\int_0^{2\pi} x(t)\sin kt\,dt, \qquad k = 1,2,\dots,m. \tag{6}$$

We call $x_m \in E_m$ a Galerkin approximation [of a 2π-periodic solution of (5)] of order m, if x_m satisfies the Galerkin equations

$$\dot{x}_m = -\frac{\lambda}{\omega}\mathcal{P}_m \mathcal{F}\mathcal{T}_\omega x_m. \tag{7}$$

This is a system of $2m+1$ nonlinear equations for $2m+1$ unknown coefficients. Its explicit form can easily be derived using (6).

We remark that only formally is (7) not given in the proper form of the Galerkin equations (see [8]): One can use an integrated version of (5) which gives rise to a completely continuous integral operator $\mathcal{M}: C_{2\pi} \to C_{2\pi}$ whose set of fixed points $x = \mathcal{M}x$ coincides with the 2π-periodic solutions of (5) [10,20]. It is easy to see that the corresponding Galerkin equation $x_m = \mathcal{P}_m \mathcal{M} x_m$ is equivalent to the system (7). Thus, the general convergence properties for Galerkin methods [8] also apply to (7).

But (7) is not yet sufficient for the computation of periodic solutions, since (5) is autonomous and we do not know the exact frequencies $\omega = 2\pi/T$. Moreover, periodic solutions and their Galerkin approximations are not isolated, which poses an additional problem. The remedy for these problems is to consider either λ or ω as an unknown variable and to enlarge the system of Galerkin equations (7) by an extra equation. In our case the periodic solutions are oscillating around 0, and thus

$$x_m(0) = 0 \tag{8}$$

is suitable. In terms of the coefficients of x_m this is

$$\tfrac{1}{2}a_0 + a_1 + \cdots + a_m = 0,$$

which should be used to eliminate one of the unknown coefficients.

For the approximation of S-solutions or special S-solutions we can reduce the system (7), (8) considerably. This is due to the symmetry of these solutions, which is also reflected in Equation (5) and which carries over to (7), (8). As a result, for the actual computation of approximations of S-solutions we may double the order m of the Galerkin equations. For special S-solutions we may even quadruple m.

In the numerical evaluation of the expressions in the system (7), (8) the only apparent problem is the computation of $\mathcal{T}_\omega x_m$ from the given coefficients of x_m and the computation of the Fourier coefficients of $\mathcal{F}\mathcal{T}_\omega x_m$. This can be achieved most efficiently by the fast-Fourier-transform methods. Two transformations are necessary, each costing $O(m \log m)$ operations. If one intends to solve the nonlinear system (7), (8) by Newton's method, derivatives are required. In the case that f is differentiable explicit formulas are given in [20]. Their numerical evaluation requires only two Fourier transformations and another $12m^2$ operations, which seems a small effort in comparison with the numerical approximation of the derivative of a translation operator as it is carried out in [4].

For the computation of bifurcation diagrams we need a continuation method. Here we employ a predictor-corrector algorithm based on a piecewise linear approximation of the underlying mappings. It is called SCOUT (Simplicial Continuation Utilities) and has been developed in [6,19,20]. Any other path-following method may be used, provided that it is able to resolve singularities such as turning points and bifurcations.

We conclude this section with the observation that conjugacy of S-solutions and also bifurcation of periodic solutions is maintained by the associated Galerkin approximations.

PROPOSITION 10. *Let $x_m \in E_m$ be a Galerkin approximation of an S-solution of* (5), *i.e., x_m solves the system* (7), (8) *and $x_m(t+\pi) = -x_m(t)$ for all $t \in \mathbb{R}$. Set $a = \pi/\omega - 1$ and define*

$$\tilde{\lambda} = a\lambda, \qquad \tilde{\omega} = a\omega,$$

$$\tilde{x}_m(t) = -x_m(-t) \qquad \text{for} \quad t \in \mathbb{R}.$$

Then $\tilde{x}_m \in E_m$ is a Galerkin approximation of an S-solution of (5) *with λ and ω replaced by $\tilde{\lambda}$ and $\tilde{\omega}$.*

PROOF. Let $a_0, \ldots, a_m$ and $b_1, \ldots, b_m$ be the Fourier coefficients of x_m. Then $a_k = b_k = 0$ for all even indices k, and

$$\tilde{x}_m(t) = \sum_{\substack{k=1 \\ k \text{ odd}}}^{m} -a_k \cos kt + b_k \sin kt.$$

Comparing coefficients, we see that it suffices to show the identities

$$\int_0^{2\pi} f(\tilde{x}_m(t-\tilde{\omega}))\begin{Bmatrix} \sin kt \\ \cos kt \end{Bmatrix} dt = \int_0^{2\pi} f(x_m(t-\omega))\begin{Bmatrix} -\sin kt \\ \cos kt \end{Bmatrix} dt$$

for odd indices k with $0 < k \leqslant m$. The odd-harmonic symmetry of $\tilde{x}_m$ and $\tilde{\omega} = \pi - \omega$ yields $\tilde{x}_m(t-\tilde{\omega}) = x_m(2\pi - \omega - t)$, which implies the above integral equations.

As a consequence of Theorem 2 or Proposition 6(ii) we have that a continuum of S-solutions which contains special *and* nonspecial S-solutions must have a bifurcation point. Proposition 10 reveals that the analogue for Galerkin approximations of S-solutions is also true.

The same conclusion holds for bifurcation of S^{2k}-solutions from S^k-solutions, which for $k \geqslant 2$ is a period doubling bifurcation: A fixed point $\varphi \in P$, $\varphi = \mathbb{S}_\lambda^{2k}(\varphi)$, implies that $\psi = \mathbb{S}_\lambda^k(\varphi) \in P$ is also a fixed point of $\mathbb{S}_\lambda^{2k}$, and one can easily check that the Galerkin approximations of S^{2k}-solutions have the corresponding property. This implies the above statement.

From Theorem 2 and Proposition 6(i) we conclude that there is a bifurcation of periodic solutions from the trivial stationary solution at

$$\lambda_k = \frac{(4k+1)\pi}{2f'(0)}, \qquad k = 0,1,2,\ldots.$$

Since the corresponding "eigenfunctions" $\sin[(4k+1)t]$ of the linearized equation (5) are elements of the subspaces E_{4k+1}, we can conclude that these bifurcations also take place in the Galerkin approximations of the order $4k+1$ [20].

The above discussion shows that our Galerkin scheme preserves some important features of the global structure of periodic solutions of the differential equation (3) or (5). This is in contrast to other methods such as the discretization of the shift operator.

4. NUMERICAL RESULTS

In the first series of experiments we use the Galerkin scheme of Section 3 to approximate S-solutions. The order of the approximation is $m = 30$ or $m = 40$. Using the continuation algorithm SCOUT, we compute the *global* bifurcation diagrams. In order to gain *local* accuracy for a specific solution we use an approximation given by SCOUT as a starting guess for Newton's method applied to the Galerkin equations of order $m = 98$.

4.1. S-solutions for Yorke's Equation

Figure 5 shows the bifurcation diagram for S-solutions of Yorke's equation (2) for the case $p = 8$. It is more instructive to view the global solution structure in the (λ, T) plane, T being the period of the solutions (Figure 6).

A few of the S-solutions for $p = 8$ are given in Figure 7. We remark that, as expected, all the plots in Figure 6 reflect the conjugacy relation of S-solutions discussed in Section 2. Let us summarize our numerical results in the following conjecture (only for S-solutions with a period $T > 4$; compare Figure 8).

CONJECTURE 11. For $\dot{x}(t) = -\lambda x/(1+|x|^p)$, $2 \leqslant p \leqslant 8$, we have:

(i) On the continuum of special S-solutions there exists a bifurcation point for S-solutions.

(ii) The bifurcating S-solutions with periods $T > 4$ form a continuum K_1, which is unbounded in λ and contains a turning point.

(iii) There exists a disjoint continuum K_2 of S-solutions of periods $T > 4$ which is also unbounded in λ and contains a turning point.

(iv) The continua K_1 and K_2 can be parametrized above the period T. The periods belonging to K_1 and K_2 are disjoint intervals $(4, T_1)$ and (T_2, T_3) where $T_3 \leqslant \infty$ and the values of T_1, T_2, and T_3 depend on the exponent p.

(v) There are no S-solutions with period T, $T_1 \leqslant T \leqslant T_2$.

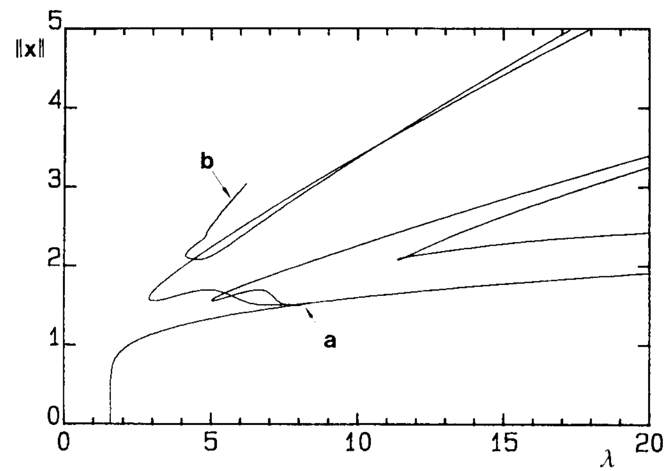

FIG. 5. Complete bifurcation diagram of S-solutions of Yorke's equation (2) with $p = 8$. There is only one bifurcation point (labelled "a") at $\lambda \approx 8.2$ on the continuum of special S-solutions. The branch which carries the label "b" consists of S-solutions where the period T tends to ∞ for growing parameters λ. Therefore it is graphed only for $\lambda \leqslant 6$. (At $\lambda = 6$ we have $T \approx 20$).

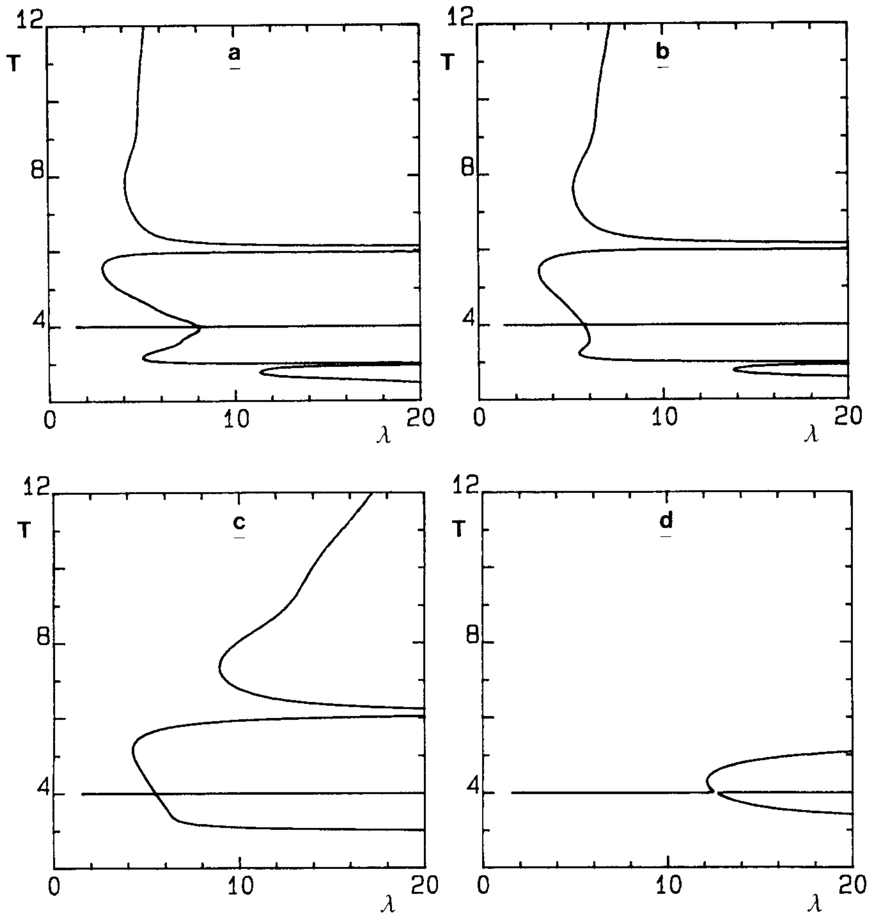

FIG. 6. Bifurcation diagrams of S-solutions of (2): (a) $p = 8$, (b) $p = 6$, (c) $p = 4$, (d) $p = 2$. As p decreases the existence of S-solutions is limited to larger and larger values of λ.

REMARKS.

1. *For* $p > 3$ *there are S-solutions with very large periods. These are the solutions which have to exist due to Theorem* 4.

2. *In* [3] *Equation* (2) *is used as a test example for detecting and resolving secondary bifurcations in connection with a derivative-free arc continuation method. There a bifurcation of* S^2*-solutions is encountered* (*see Section* 4.5); *however, the bifurcation of S-solutions* (i) *is missed.*

3. *In* [16, *Remark* 2.2] *it is conjectured that for* $1 < p < 3$ *all S-solutions of* (2) *are special S-solutions. But Figure* 6(*d*) *shows numerical evidence that*

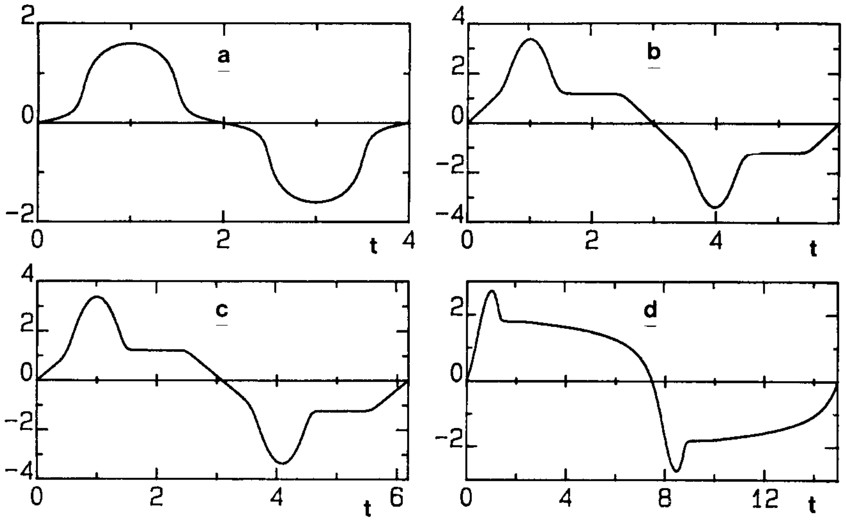

FIG. 7. S-solutions of (2) for $p = 8$:

Part	Type of solution	λ	T	$\|x\|$
(a)	Special S-solution	10	4.000	1.599
(b)	S-solution in K_1	10	5.966	3.383
(c)	S-solution in K_2	10	6.183	3.371
(d)	S-solution in K_2	5.496	14.951	2.723

nonspecial S-solutions do exist. Thus, the uniqueness of slowly oscillating periodic solutions at $p = 1$ (Theorem 3) does not carry over to parameters $p > 1$.

4.2. Transition to Uniqueness. I

Let us investigate the last remark a little more closely. We compute the bifurcation diagrams of S-solutions of (2) for parameters p that approach $p = 1$ from above. Here it is necessary to allow large values of λ, say $\lambda \leqslant 100$. We find that for $\lambda \leqslant 100$ and $p \leqslant 1.48$ only special S-solutions exist. For $1.5 \leqslant p \leqslant 2.0$ the continua K_1 and their conjugates are given in Figure 9. For $\lambda \leqslant 100$ S-solutions of the continua K_2 exist only if $p \geqslant 2.5$.

Thus, as $p \to 1$ the transition to uniqueness occurs in the following way: For a given $p > 1$ let λ_p denote the infimum of the parameter values of λ for which there exist nonspecial S-solutions of (2). Then $p \to 1$ implies $\lambda_p \to \infty$.

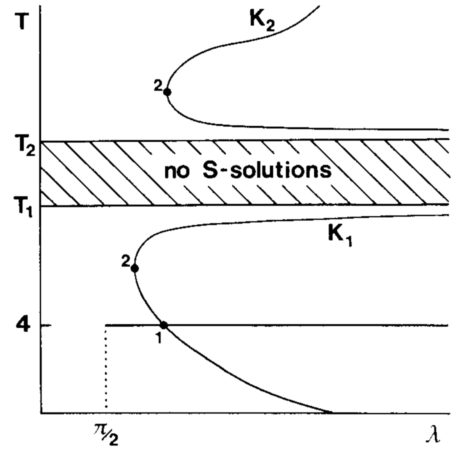

FIG. 8. Schematic bifurcation diagram of S-solutions illustrating Conjecture 11. 1: bifurcation point; 2: turning points.

4.3. *Connecting Disjoint Branches of S-solutions*

In this section we focus on parts (iv) and (v) of Conjecture 11. Some of our earlier numerical studies seemed to suggest (see [16, Remark 1.3]), that 6 is a crucial period in the sense that $T_1 = T_2 = 6$. However, from the results in Section 4.1 for $p = 8$ we estimate $T_1 = 5.96\ldots$ and $T_2 = 6.11\ldots$. Moreover, for $p = 4$ and $\lambda = 20$ we compute a Galerkin approximation x_m, $m = 98$, of an

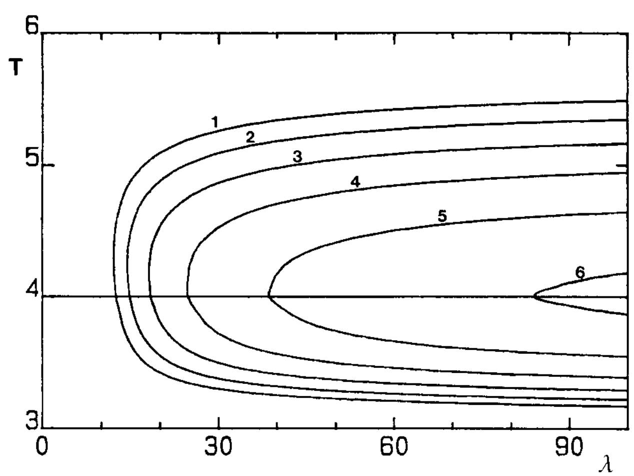

FIG. 9. Bifurcation diagram of S-solutions of (2). Branch 1: $p = 2.0$; 2: $p = 1.9$; 3: $p = 1.8$; 4: $p = 1.7$; 5: $p = 1.6$; 6: $p = 1.5$. The order of the Galerkin scheme is $m = 40$.

S-solution of the continuum K_1 which has period $T = 6.043$. Since K_1 links this solution with a special S-solution of period 4, we may conclude that there exists an S-solution in K_1 for $\lambda < 20$ which has the exact period 6. The defect $\|\dot{x}_m + (\lambda/\omega)\mathcal{F}\mathfrak{T}_\omega x_m\|$ of the mentioned solution is only about 10^{-4}, and thus our conclusion should be numerically sound. Therefore we no longer believe that 6 is such a crucial period.

In Figure 7(b, c) we have S-solutions of K_1 and K_2 for $\lambda = 10$ which have periods T close to T_1 and T_2. The two solutions are very similar. Both contain an interval of length ≈ 1 where they have almost zero slope. This is possible only because $f(x) \to 0$ as $|x| \to \infty$. We conjecture that it is this property of f which is responsible for $T_1 \leqslant T_2$, or for the fact that K_1 is disjoint from K_2. In order to perform a test we consider the 2-parameter problem

$$\dot{x}(t) = -f_{\lambda,\varepsilon}(x(t-1)), \qquad \lambda, \varepsilon > 0,$$
$$f_{\lambda,\varepsilon}(x) = \frac{\lambda x}{1 + x^8} + \varepsilon\,\mathrm{sign}(x)\left(1 - e^{-|x|}\right)^2. \tag{9}$$

For $\varepsilon = 0$ this coincides with the case $p = 8$ of (2) and we have $f_{\lambda,\varepsilon}(x) \to \pm\varepsilon$ as $x \to \pm\infty$. We experiment with $\varepsilon = 0.1, 0.05, 0.02, 0.01$. In fact, for all of these values there exists only one continuum of S-solutions with periods greater than 4 (see Figure 10). As $\varepsilon \to 0$ we approach the situation of Figure 6(a).

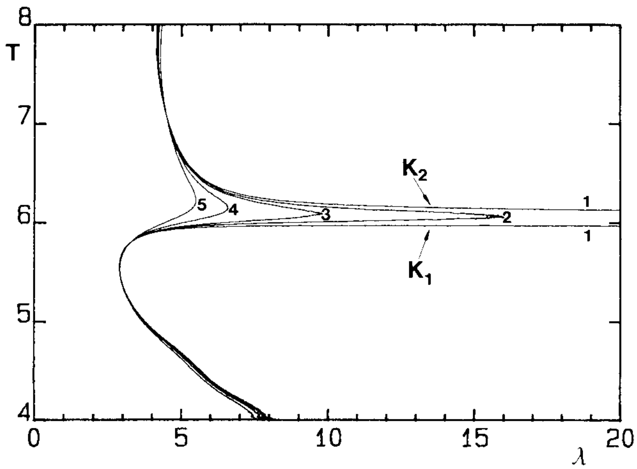

FIG. 10. Lii ng f links continua K_1 and K_2. Curve 1: $\varepsilon = 0$; 2: $\varepsilon = 0.01$; 3: $\varepsilon = 0.02$; 4: $\varepsilon = 0.05$; 5: $\varepsilon = 0.1$.

4.4. *Transition to Uniqueness. II*

Instead of changing p in $f(x)=x/(1+|x|)^p$ from $p=8$ to $p=1$, we now consider the linear, convex homotopy

$$f_\varepsilon(x)=(1-\varepsilon)\frac{x}{1+x^8}+\varepsilon\frac{x}{1+|x|}, \qquad 0\leqslant\varepsilon\leqslant 1,$$

for the delay equation

$$\dot{x}(t)=-\lambda f_\varepsilon(x(t-1)). \tag{10}$$

We have that f_ε satisfies (H) and $f_\varepsilon(x)\to\pm\varepsilon$ as $x\to\pm\infty$. In the case $\varepsilon>0$ it follows from a result in [13] that the periods of S- and S^2-solutions of (10) are bounded and must tend to 4 as $\lambda\to\infty$. Surprisingly, we have for sufficiently large parameters λ that these periods are actually equal to 4. This furnishes a transition to uniqueness of slowly oscillating periodic solutions as $\varepsilon\to 1$, which is completely different from the one in Section 4.2 (see Figure 11).

As expected from the results in Section 4.3, for $\varepsilon>0$ we have only one continuum of nonspecial S-solutions with periods greater than 4. The new phenomenon here is that this branch is not unbounded, but leads to another secondary bifurcation on the continuum of special S-solutions. As ε grows, we observe that this "loop" of S-solutions shrinks down to a point at approximately $\lambda=6$. Already at $\varepsilon=0.5$ we have uniqueness of S-solutions. From Figure 11(b, c) we conclude that the same fate applies to the S^2-solutions: For $\varepsilon=0.1$ they form two loops, one bifurcating from special S-solutions and the other one bifurcating from S-solutions with periods less than 4. These loops have disappeared by the time ε reaches 0.4 or 0.3, respectively.

Also note that the statement about the "slope" of the backward bifurcation of S-solutions in Lemma 9 is clearly reflected in all graphs of Figure 11.

4.5. *S^2- and S^4-solutions*

We now consider S^2- and S^4-solutions of (2) for $p=8$. For their approximation we use the Galerkin scheme of order $m=15$, i.e., we have to deal with 32 unknowns. We distinguish between S^2-solutions that bifurcate from

(1) special S-solutions,
(2) S-solutions of K_1,
(3) S-solutions of K_2.

Let us briefly describe these solutions, which are pictured in the perhaps confusing bifurcation diagram of Figure 12.

(1): There are three such continua, bifurcating at $\lambda=2.0$, $\lambda=3.5$, and $\lambda>20$. The first of these S^2-solutions we call S^2-solutions of type A [see

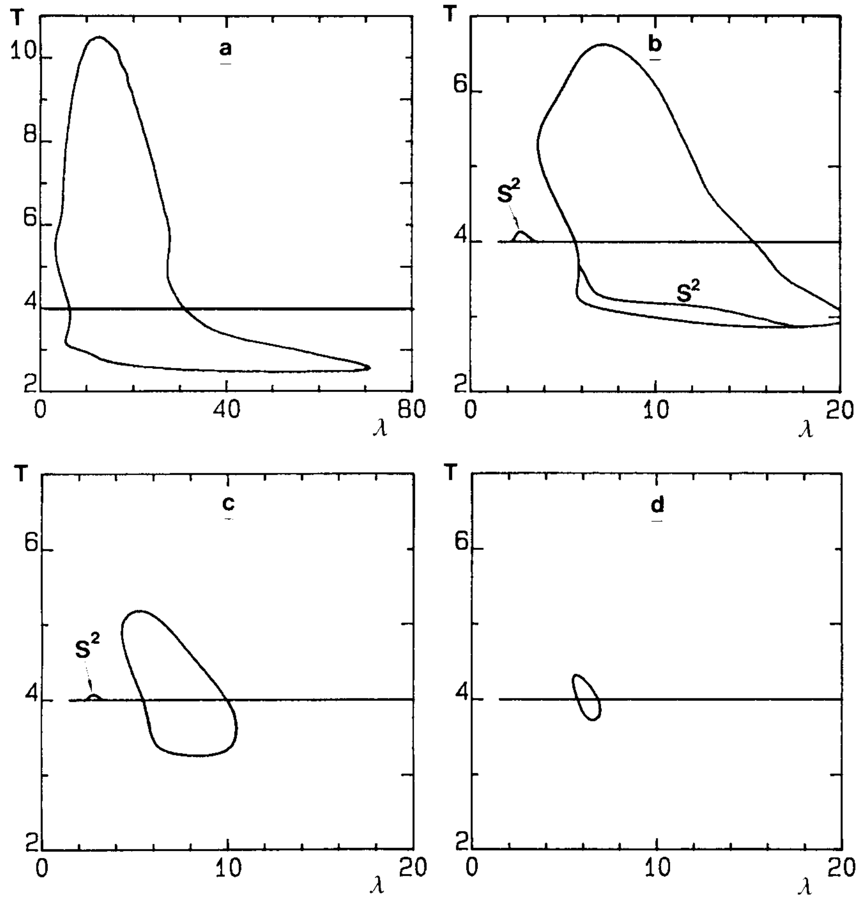

FIG. 11. Bifurcation diagrams of S-solutions for (10): (a) $\varepsilon = 0.1$, (b) $\varepsilon = 0.2$, (c) $\varepsilon = 0.3$, (d) $\varepsilon = 0.4$. In (b) and (c) S^2-solutions are also given. The order of the Galerkin scheme is $m = 30$ for (a) and $m = 15$ for (b), (c), and (d).

Figure 13(a)]. They have another secondary bifurcation point at $\lambda = 2.5$ where S^4-solutions emanate [see Figure 13(b)]. These bifurcations correspond to the cascading bifurcation of Conjecture 4 (compare [6]). The second branch of S^2-solutions is not unbounded, but it returns to the continuum of special S-solutions.

(2): We have found one branch of S^2-solutions, which we call S^2-solutions of type B. Again there is a branch of S^4-solutions bifurcating from these S^2-solutions. Apparently there is another cascading bifurcation starting out

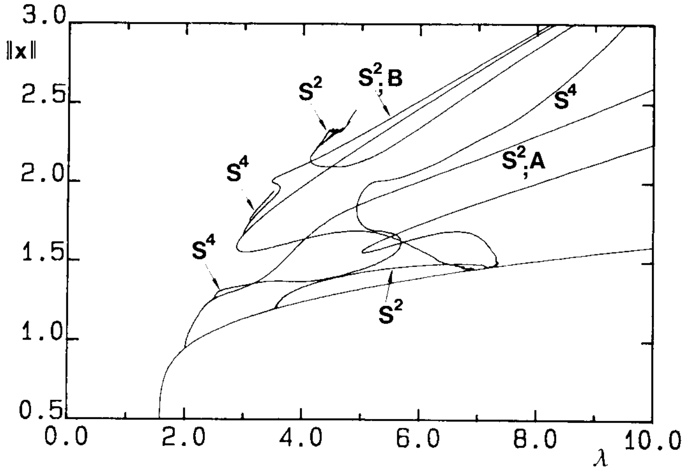

FIG. 12. Bifurcation diagram of S-, S^2-, and S^4-solutions of (2) for $p = 8$.

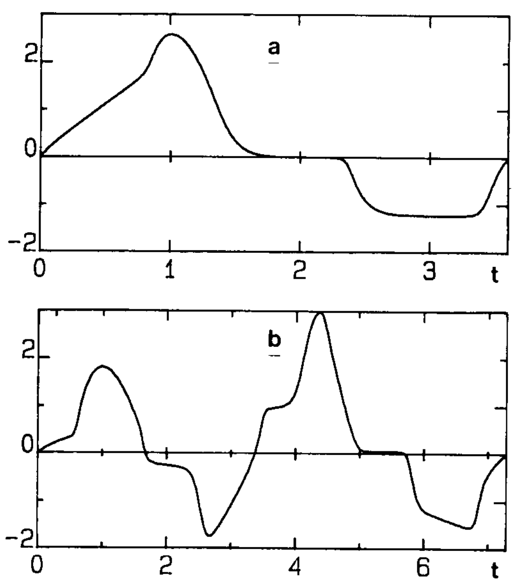

FIG. 13. (a) S^2-solution of type A, $\lambda = 10$, period $T = 3.601$, $m = 49$. (b) S^4-solution at $\lambda = 10$, period $T = 7.280$, $m = 49$.

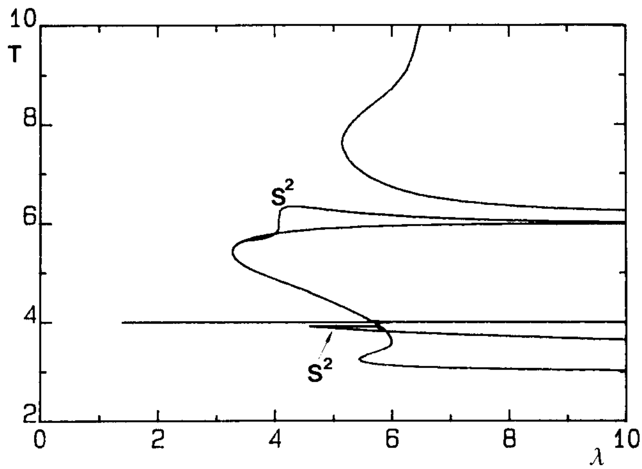

FIG. 14. Multiple bifurcation point for $p = 6$.

from an S-solution of K_1. For large parameters λ the shape of these S^2- and S^4-solutions approaches the shape of the S-solutions of K_1 [see Figure 7(b)].

(3): There are two bifurcation points, which are linked by a continuum of S^2-solutions.

As we decrease the parameter p, something interesting happens to the S^2-solutions of type A. The secondary bifurcation point "moves forward" until for approximately $p = 6$ it coincides with the bifurcation point of S-solutions (see Figure 14). For yet smaller p we find that the bifurcation of S^2-solutions of type A no longer happens at a special S-solution, but it starts out from an S-solution with a period $T < 4$, i.e. a solution conjugate to an S-solution of K_1.

For $p < 6$ we thus conjecture that with increasing λ we first encounter a secondary bifurcation on the continuum of special S-solutions. There two conjugate branches of S-solutions emanate. On both of these branches we then have a cascading bifurcation of S^2-, S^4-, S^8-,... solutions.

4.6. *Cascading Bifurcation*

Consider Equation (2) with $p = 8$. For a given parameter λ in the range where we assume the cascading bifurcation of Conjecture 4, we expect an arbitrarily chosen initial function $0 \neq \varphi_0 \in P$ to yield an asymptotically periodic sequence $\mathbb{S}_\lambda^i(\varphi_0)$, $i = 1, 2, \ldots$, which determines a unique S^k-solution. In order to compute this (minimal) number k, we first perform j iterations of the

TABLE 1
THE FIRST 6 BIFURCATION POINTS

k	λ_k
1	2.0026
2	2.5179
3	2.6238
4	2.6693
5	2.6809
6	2.6831

shift operator, so that transients die out. Then we compute

$$e_i = \left\| \mathcal{S}_\lambda^{2^i}(\varphi^*) - \varphi^* \right\|, \qquad i = 0, 1, 2, \ldots,$$

where we have set $\varphi^* = \mathcal{S}_\lambda^j(\varphi_0)$. We define *a priori* a tolerance tol > 0 and conclude that φ^* induces an S^q-solution with $q = 2^k$ if $e_i \leqslant$ tol for $i = k, k+1, \ldots$ and $e_i >$ tol for $i = 0, 1, \ldots, k-1$. By repetition of this procedure for various parameters λ we can estimate the bifurcation parameters λ_k.

For the numerical implementation a discretization of the shift operator $\mathcal{S}_\lambda$ is necessary. We use an equidistant discretization of the time t (n subintervals per unit) and the trapezoidal rule for the integration of the initial-value problem (4). The first zero z_1 and the dicrete version of $\mathcal{S}_\lambda(\varphi)$ are then obtained by a linear interpolation.

We choose $n = 50$, $j \approx 1000$, and tol $= 0.00005$. For the first initial function φ_0 we select $\varphi_0(t) = \cos(\pi t/2)$, and thereafter we use the result φ^* of the previous iteration. The experiment was successful for the first 6 secondary bifurcations; the results are given in Table 1.

Thus, our numerical computation confirms the conjecture about the cascading bifurcation. However, we remark that the exact values of the numbers λ_k depend on the chosen approximation scheme.

5. CONCLUDING REMARKS

In Section 1 we have summarized some of the rigorous results dealing with the bifurcation structure of the nonlinear differential delay equation under consideration here. Our numerical exposition confirms these results and also shows a wealth of new phenomena. We hope that this paper stimulates their further investigation.

One interesting result along this line has recently been obtained by Walther [21]: He has proved, for a certain class of nonlinearities, that there is a secondary bifurcation of periodic solutions taking place on the continuum of special S-solutions.

This paper is only a case study of a set of very special delay equations. It would be interesting to extend this study to other delay equations. For instance, one could consider.

(1) nonlinearities with several zeroes,
(2) nonlinearities that are not odd,
(3) equations with several delays or distributed delay,
(4) systems of such equations,
(5) periodically forced delay equations.

I thank H.-O. Peitgen for raising my interest in delay equations and for many encouraging discussions on this topic.

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