CHARACTERIZATION OF PERIODIC SOLUTIONS OF SPECIAL DIFFERENTIAL DELAY EQUATIONS

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1. Introduction

The aim of this paper is to derive a simple finite dimensional characterization of periodic solutions of the differential delay equation

$$(1) \qquad \dot{x}(t) = -\lambda f(x(t-1)), \quad \lambda > 0$$

where $f : R \to R$ is an odd and piecewise constant function satisfying $xf(x) > 0$ for all $x \neq 0$. Here a solution x of (1) is a continuous piecewise linear function which solves the integrated version

$$(2) \qquad x(t) = x(0) - \lambda \int_{-1}^{t-1} f(x(s))\, ds$$

of (1). From our results we obtain a suitable numerical procedure for the computation of periodic solutions of (1) and we report its performance on a test example.

Our task is motivated by the study of periodic solutions of (1) where f is a continuous nonlinearity, e. g. $f(x) = x/(1+x^8)$ (see [1, 3,6,7]). Recently H. Peters [4] has modelled this nonlinearity by a piecewise constant function which has two steps (for $x > 0$). He was able to completely and explicitly compute all periodic solutions. We will modify and extend his approach for nonlinearities f with n steps. Thus, we may use a better piecewise constant approximation to a continuous nonlinearity. However, we cannot expect to be able to explicitly compute nontrivial periodic solutions, because the ~~the~~ computational complexity is too great even if n is of moderate size, say $n \geq 4$. Instead we have to rely on the computer which may numerically solve the problem.

Our approach has one promising aspect, namely that a solution to (1) is piecewise linear and therefore may be computed exactly up to round off errors. Hence, there are no discretization errors involved! We may say that the discretization has already occured in the choice of the piecewise constant nonlinearity f .

We now scetch one the standard procedures of how the problem of computing periodic solutions may be cast into an operator equation. Let $\varphi \in C[-1,0]$ denote a continuous initial function for the initial value problem

$$(3) \qquad \begin{cases} \dot{x}(t) = -\lambda f(x(t-1)) & \text{for} \quad t \geq 0 , \\ x(t) = \varphi(t) & \text{for} \quad -1 \leq t \leq 0 . \end{cases}$$

There is a unique solution x_φ of (3) defined on the interval $[-1,\infty)$. We call a solution x of (1) or x_φ of (3) *slowly oscillating* if it has infinitely many zeroes and if the distance between any two zeroes is greater than 1 . For the study of slowly oscillating, periodic solutions of (1) we may restrict to initial functions from the set

$$P = \{\varphi \in C[-1,0] \mid \varphi(-1) = 0 , \varphi \text{ is strictly monotonically increasing}\} .$$

Definition 1

For $\varphi \in P$ let x_φ denote the corresponding solution of (3) . Then there is a zero $z_1 > 0$ of x_φ such that $x_\varphi(t) > 0$ for $-1 < t < z_1$. The *shift operator* $S_\lambda : P \to P$ is defined by $S_\lambda(\varphi) : t \to -x_\varphi(z_1+1+t)$.

Since f is odd we have that a fixed point of the shift operator S_λ or of one of its iterates S_λ^k , $k = 2,3,\ldots$ induces a slowly oscillating, periodic solution of (1) , which we call an *S-solution* or *S^k-solution* respectively. Moreover, there exist *special S-solutions* x of (1) which have an additional symmetry : If z denotes a zero of x , then we have $x(z+t) = x(z+2-t)$ for all $t \in R$. Thus, these solutions are sinoidal and they have the period 4 .

Note that the nonlinearity f is only piecewise constant and thus, the shift operator S_λ is not continuous. However, in the next section we will see how the piecewise constant structure of f facilitates a modified, continuous, and even finite dimensional shift operator.

2. The shift operator for equivalence classes of initial functions

For $\varphi_1,\varphi_2 \in P$ let $\varphi_1 \sim \varphi_2$ if and only if $x_{\varphi_1}(t) = x_{\varphi_2}(t)$ for $t \geq 0$. "$\sim$" is an equivalence relation. For continuous nonlinearities

f in (1) we have in general that the equivalence classes $[\varphi]$ trivially contain only one element, namely φ itself. But with piecewise constant nonlinearities $[\varphi]$ may have many elements. We show that the set of all equivalence classes $P/\sim$ is a finite dimensional set. Let us first define a suitable set of nonlinearities.

Definition 2

For $n \in \{1,2,...\}$ let F_n denote the set of real functions satisfying

(a) $f(-x) = -f(x)$ for $x \in R$.

(b) $xf(x) > 0$ for $x \neq 0$.

(c) There exist numbers $f_1,...,f_n > 0$ and a subdevision $0 = x_0 < x_1 < ... < x_n = \infty$ such that for $k = 1,...,n$ $f(x) = f_k$ holds if $x \in J_k = (x_{k-1},x_k]$.

(d) $f_k \neq f_{k+1}$ for $k = 1,...,n-1$.

We say that a function $f \in F_n$ has n steps. The following mapping I_f extracts the necessary information from an initial function $\varphi \in P$, which is relevant for the integration of (3) .

Definition 3

Let $I_f = (I_f^0,...,I_f^n) : P \to R \times R^n$ be defined by

(a) $I_f^0(\varphi) = \varphi(0)$.

(b) Let $x_0,...,x_n$ be as in Definition 2 and assume $\varphi(0) \in J_j = (x_{j-1},x_j]$.

(c) If $j = 1$, then set $I_f^1(\varphi) = 1$ and $I_f^k(\varphi) = 0$ for $k = 2,...,n$.

(d) If $j > 1$, then set

$$I_f^1(\varphi) = \varphi^{-1}(x_1) + 1 ,$$
$$I_f^k(\varphi) = \varphi^{-1}(x_k) + 1 - \sum_{i=1}^{k-1} I_f^i(\varphi) \quad \text{for } k = 2,...,j-1 ,$$
$$I_f^j(\varphi) = 1 - \sum_{i=1}^{j-1} I_f^i(\varphi) ,$$
$$I_f^k(\varphi) = 0 \quad \text{for } k = j+1,...,n .$$

Figure 1 illustrates the definition of I_f . Let $D = I_f(P)$. Obviously we have $D \subset R^+ \times \Delta_{n-1}$ where Δ_{n-1} denotes the standard $(n-1)$-simplex in R^n . The following lemma states that we can identify $P/\sim$ with the n-dimensional range of D of I_f .

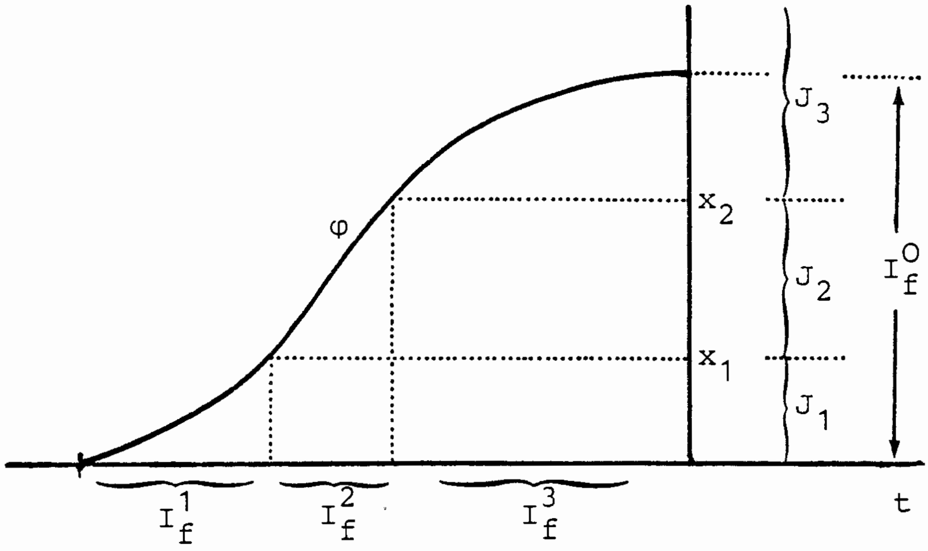


Figure 1 The definition of I_f

Lemma 4

Let $f \in F_n$, $\varphi_1, \varphi_2 \in P$. Then

$$\varphi_1 \sim \varphi_2 \quad \Leftrightarrow \quad I_f(\varphi_1) = I_f(\varphi_2) \ .$$

Proof: Assume $\varphi_1 \sim \varphi_2$ and that $k = \min\{k \mid I_f^k(\varphi_1) \neq I_f^k(\varphi_2)\}$ exists. If $k = 0$ then $\varphi_1(0) \neq \varphi_2(0)$ contradicting $x_{\varphi_1}(0) = x_{\varphi_2}(0)$. Otherwise we have for sufficiently small $\varepsilon > 0$ the contradiction $x_{\varphi_1}(t_0+\varepsilon) \neq x_{\varphi_2}(t_0+\varepsilon)$ where

$$t_0 = \sum_{i=1}^{k-1} I_f^i(\varphi_1) + \min\{I_f^k(\varphi_i) \mid i = 1,2\} \ .$$

If we assume $I_f(\varphi_1) = I_f(\varphi_2)$, then it follows from the construction of I_f that we have $\varphi_1 \sim \varphi_2$.

It is easy to see that $D = I_f(P)$ is a connected subset of $R^+ \times \Delta_{n-1}$. If $\varphi_0, \varphi_1 \in P$, then $\{\varphi_s = (1-s)\varphi_0 + s\varphi_1 \mid s \in [0,1]\} \subset P$ and I_f is continuous on this line segment of P . Therefore $[\varphi_0]$ and $[\varphi_1]$ can be connected by the path $s \to [\varphi_s]$.

We have that $\varphi_1 \sim \varphi_2$ implies $S_\lambda(\varphi_1) = S_\lambda(\varphi_2)$. Therefore we may define an induced shift operator $\overline{S}_\lambda$ on D via

$$\begin{aligned} \overline{S}_\lambda : P/\sim &\to P/\sim \\ [\varphi] &\to [S_\lambda(\varphi)] \ . \end{aligned}$$

In contrast to the original shift operator $S_\lambda : P \to P$ we have that the induced map $\overline{S}_\lambda : P/\sim \to P/\sim$ is continuous. For the elementary but rather technical proof of this fact we refer to [6] .

Concerning slowly oscillating, periodic solutions we now have :

Corollary 5

Let $k \in 1,2,\ldots$.

(i) If $\varphi^* \in P$ such that $\varphi^* = S_\lambda^k(\varphi^*)$, then also $[\varphi^*] = \bar{S}_\lambda^k([\varphi^*])$.

(ii) If $\varphi \in P$ such that $[\varphi] = \bar{S}_\lambda^k([\varphi])$, then $\varphi^* = S^k(\varphi^*)$ where $\varphi^* = S_\lambda(\varphi)$.

Therefore, S^k-solutions of (1) are in a one-to-one correspondance with the fixed points of the n-dimensional mappings $\bar{S}_\lambda^k$. Their numerical computation is a nonlinear fixed point problem, and in our case continuation methods seem to be the most appropiate tools for this task. We employ a predictor-corrector algorithm based on a piecewise linear approximation of the underlying mapping. It is called SCOUT and has been develloped by H. Jürgens and the author (see [1,5,6]). Any other path following method may be used, provided that it is able to resolve singularities such as turning points and bifurcations. SCOUT is designed to compute the zeroes of a mapping $H : R^n \times R \to R^n$, and in order to facilitate its employment for our purpose we identify $R^+ \times \Delta_{n-1}$ with the n-dimensional nonnegative octant $[0,\infty)^n$ via $(u_0,\ldots,u_n) \to (u_0 u_1,\ldots, u_0 u_n)$ and extend $\bar{S}_\lambda^k$ to a continuous mapping $\tilde{S}_\lambda^k : R^n \to R^n$ such that the diagram

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d, "\bar{S}_\lambda^k"'] & R^+ \times \Delta_{n-1} \arrow[r] & {[0,\infty)^n} \arrow[r, hook] & R^n \arrow[d, "\tilde{S}_\lambda^k"] \\ D \arrow[r, hook] & R^+ \times \Delta_{n-1} \arrow[r] & {[0,\infty)^n} \arrow[r, hook] & R^n \end{tikzcd}$$

commutes and $\tilde{S}_\lambda^k(R^n) \subset D$. We then define

$$\begin{aligned} H : R^n \times R^+ &\to R^n \\ (u,\lambda) &\to u - \tilde{S}_\lambda^k(u) \ . \end{aligned}$$

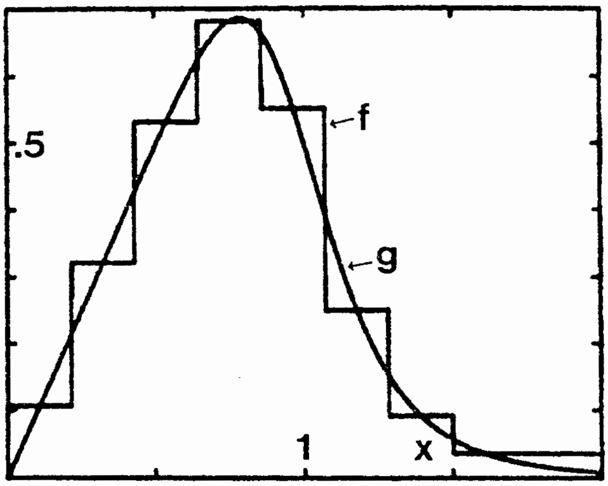
A first periodic solution as a start for the continuation method is trivially available : Let $f \in F_n$ and $x_1, f_1 > 0$ be as in Definition 2 . For $0 < \lambda < x_1/f_1$ we have that the initial function $\varphi(t) = \lambda f_1 (t+1)$ is a fixed point $\varphi = S_\lambda(\varphi)$ defining a special S-solution. Therefore $u = \bar{S}_\lambda(u)$ where $u = (\lambda f_1, 1, 0, \ldots, 0) \in D$.

If we approximate a given continuous nonlinearity by a piecewise constant function $f \in F_n$ then the equation $[\varphi] = S_\lambda([\varphi])$ is a *perturbed Galerkin equation* (in the sense of Krasnoselski´i et al [2]) for the computation of periodic solutions of the given delay equation.

3. Numerical results

All the computations were done with the SCOUT package using a triangulation with a mesh size of 0.001 . The predictor-corrector technique [5] was successfully employed such that the experiments could conveniently be done in an interactive time sharing session (on a Siemens 7.800).

In this section we are considering a special nonlinearity $f \in F_8$ designed to model the smooth function $g(x) = x/(1+x^8)$ (see Figure 2).



k	x_k	f_k
1	0.214	0.107
2	0.429	0.321
3	0.643	0.532
4	0.857	0.682
5	1.071	0.552
6	1.286	0.250
7	1.500	0.092
8	∞	0.035

Figure 2 The nonlinearity $f \in F_8$. We define $x_k = 3k/14$, $f_{k+1} = g(x_k+3/28)$ for $k = 0,\dots,7$ and $x_8 = \infty$.

Figure 3 shows a bifurcation diagram for S-solutions of (1) , where we have plotted the norm of the initial functions $\|\varphi\|$ versus λ . There is a continuum of special S-solutions emenating from the origin. The zig-

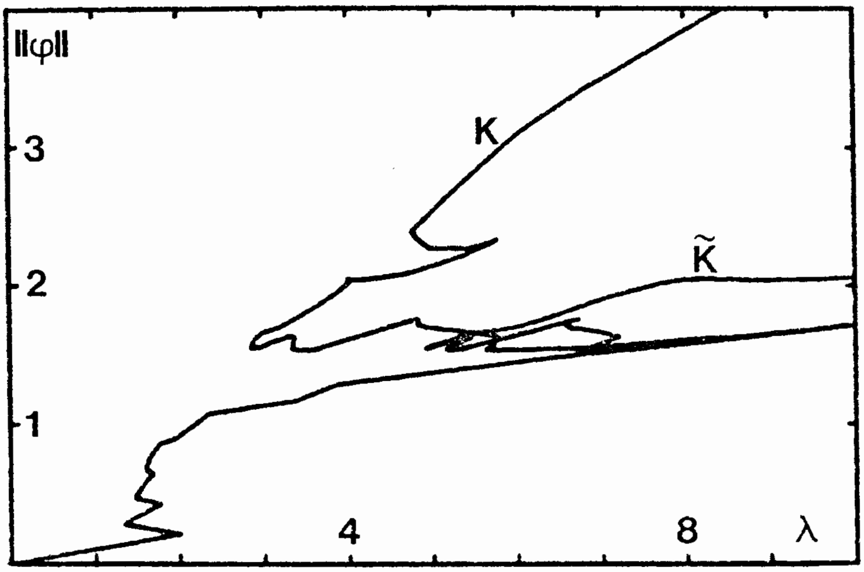


Figure 3 Bifurcation diagram of S-solutions of (1) .

zagging behaviour of this continuum for $1 < \lambda < 2$ is characteristic for equation (1) with piecewise constant nonlinearities such as f, and it has also been observed in [6,ch. 7] . At $\lambda = 12.6$ a bifurcation takes place : Two branches (K and $\tilde{K}$) of not special S-solutions bifurcate. K consists of S-solutions with periods greater than 4 whereas $\tilde{K}$ consists of S-solutions with periods less than 4 . It can be seen as in [6,7] that K and $\tilde{K}$ are conjugate in the following sense. If (x,λ) denotes an S-solution with a period T, then we have that $(\tilde{x},\tilde{\lambda})$ is also an S-solution, where $\tilde{x}(t) = -x(-t(T-2)/2)$ and $\tilde{\lambda} = \lambda(T-2)/2$. With this notation we have that $(x,\lambda) \in K$ if and only if $(\tilde{x},\tilde{\lambda}) \in \tilde{K}$.

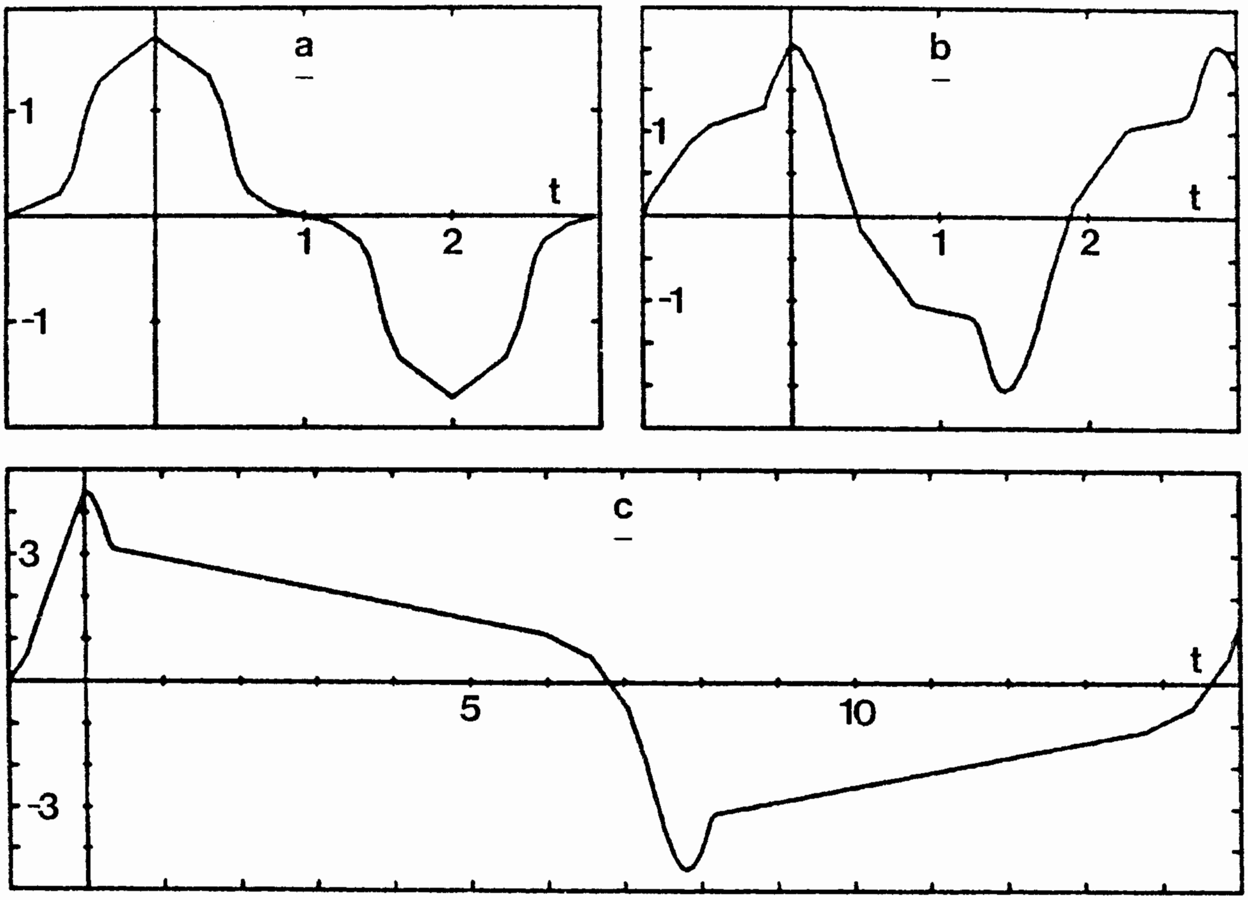


Figure 4 S-solutions of (1) for $\lambda = 10$. a) Special S-solution, b) S-solution of $\tilde{K}$ with a period $T = 2.875$, c) S-solution of K with a period $T = 15.622$. Note : For the shown initial functions φ we have $\varphi \sim S_\lambda(\varphi)$, not $\varphi = S_\lambda(\varphi)$.

Figure 4 shows three periodic solutions from the different branches at $\lambda = 10$. A stability test reveals that the S-solution of K is stable and attractive with respect to the integration of the delay equation, whereas the other two solutions are rather unstable (see Figure 5).

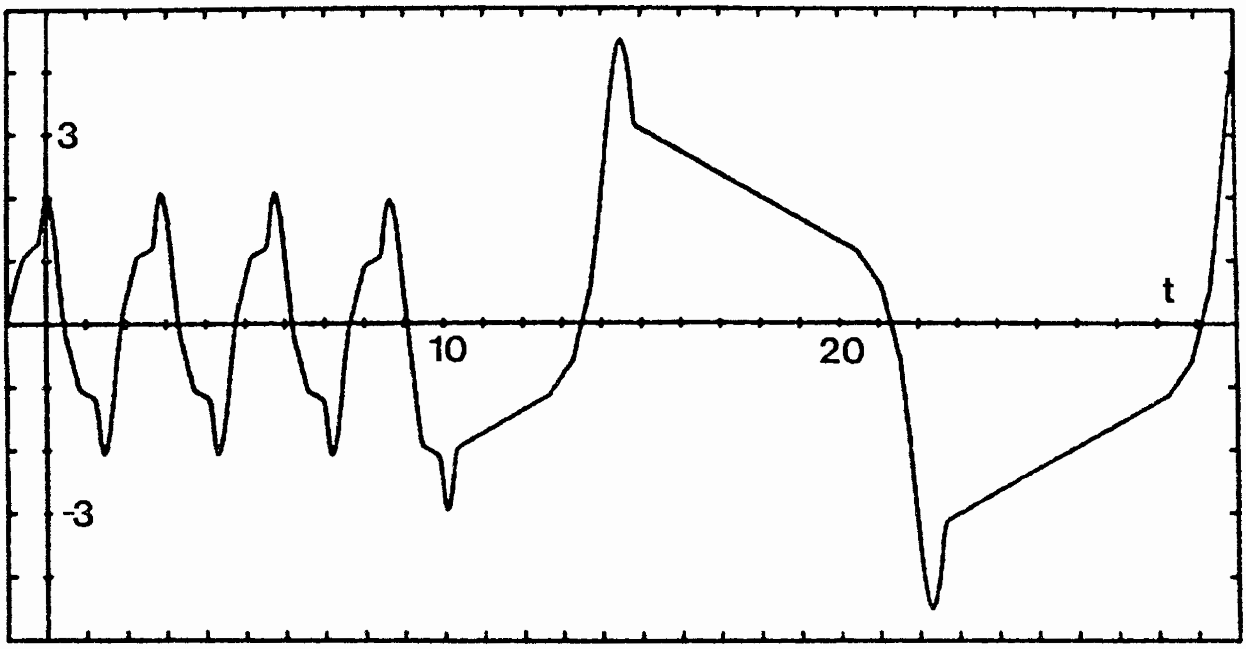


Figure 5 Transition from an S-solution of $\tilde{K}$ to an S-solution of K. The integration of (3) for the (approximate) initial function generates 3 seemingly periodic cycles and in the following an abrupt change to the S-solution of K.

The continua K and $\tilde{K}$ are not unbounded, but they meet again at a bifurcation point on the continuum of special period solutions at $\lambda = 84.95$ (see Figure 6).

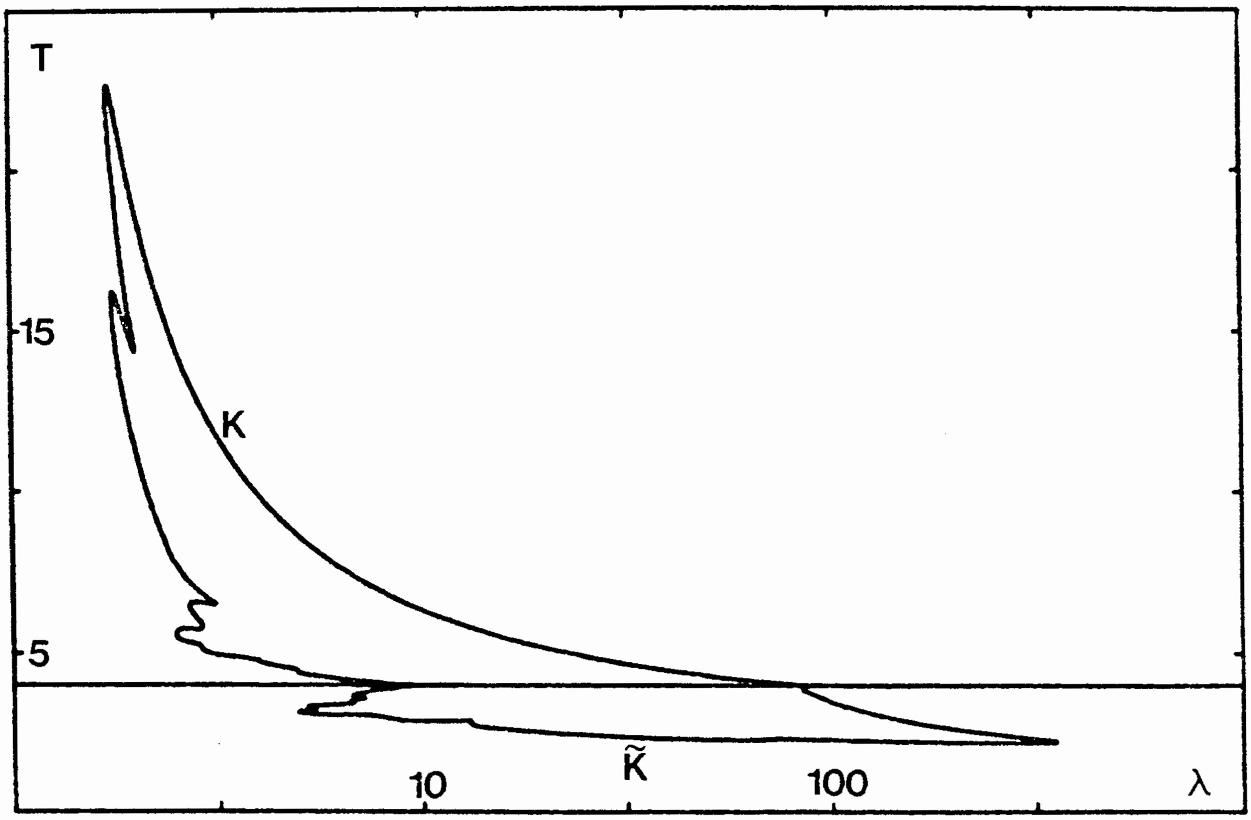


Figure 6 Bifurcation diagram of S-solutions of (1). Here the period T is graphed versus λ (in the log-scale).

In the last experiment we concentrate on S^2-solutions of (1) . There are two such continua, the first one bifurcates from special S-solutions at $\lambda = 1.79$, forms a loop, and returns to the special S-solutions at $\lambda = 1.95$. The second continuum branches off at $\lambda = 2.35$ (see Figure 7 and 8) .

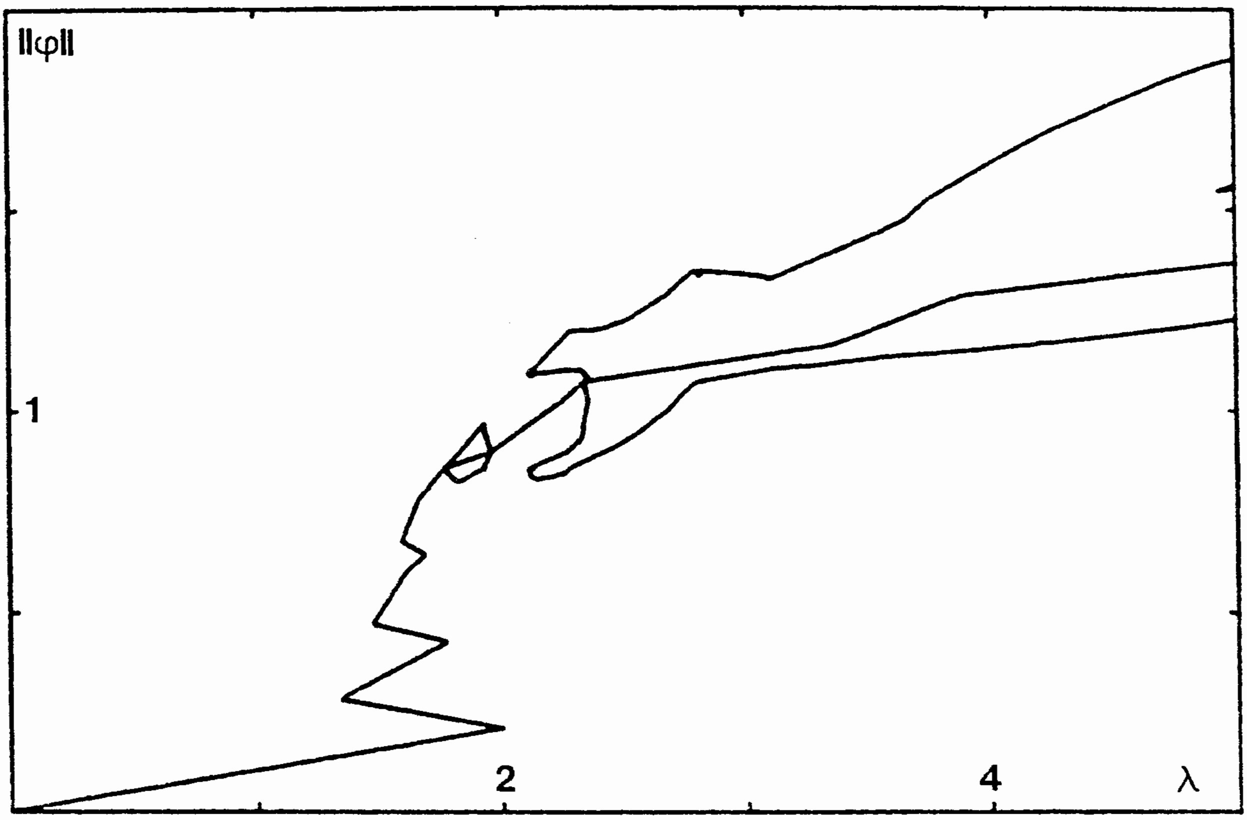


Figure 7 Bifurcation diagram of S^2-solutions of (1) .

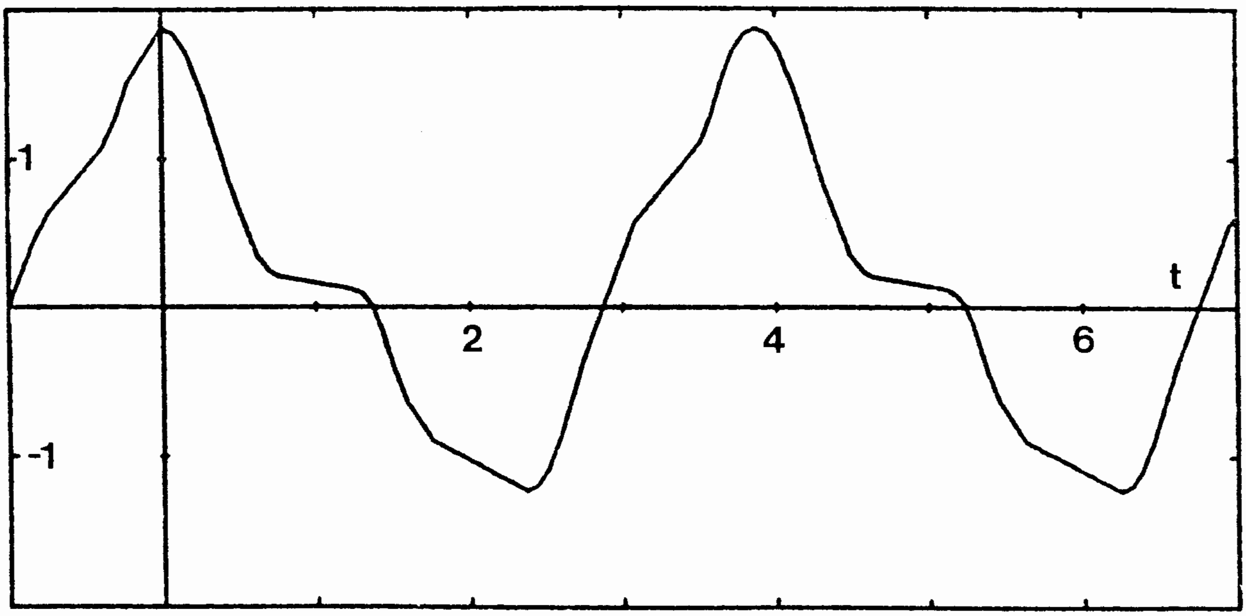


Figure 8 S^2-solution of (1) at $\lambda = 5$, the period is $T = 3.875$.

We remark that all of the above results agree with similar studies carried out in [1,6,7] for continuous nonlinearities. However, the approach of this paper as a numerical method for the computation of periodic solutions of such delay equations does not yield very satisfactory results. This is especially true for equations with nonlinearities $f \in F_n$ with a larger n, say $n \approx 20$ or for the computation of S^k-solutions with a larger k, say $k \geq 4$. This seems to de due to two facts. 1) As demonstrated, many of the periodic solutions are unstable, therefore translation operators such as S_λ are not very suitable in general. 2) The piecewise constant structure of the nonlinearities under consideration here generates many singularities (bifurcations, turning points) that are not present in equations with corresponding smooth nonlinearities.

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References

[1] H. Jürgens, H.-O. Peitgen, D. Saupe, Topological perturbations in the numerical study of nonlinear eigenvalue and bifurcation problems, in: "Analysis and computation of fixed points", S. M. Robinson (ed.), Academic Press, New York, 1980, pp. 139-182.

[2] M. A. Krasnosel´skii et al, Approximate solution of operator equations, Wolters-Noordhoff, Groningen, 1972.

[3] R. D. Nussbaum, Periodic solutions of nonlinear autonomous functional differential equations, in: "Functional differential equations and approximation of fixed points", H.-O. Peitgen, H. O. Walther (eds.), Springer Lecture Notes, Berlin, 1979, pp. 283-325.

[4] H. Peters, Globales Lösungsverhalten zeitverzögerter Differentialgleichungen am Beispiel für Modellfunktionen, Dissertation, Universität Bremen, Bremen 1981.

[5] D. Saupe, On accelerating PL continuation algorithms by predictor corrector methods, Mathematical Programming 23 (1982), pp. 87-110.

[6] D. Saupe, Beschleunigte PL-Kontinuitätsmethoden und periodische Lösungen parametrisierter Differentialgleichungen mit Zeitverzögerung, Dissertation, Universität Bremen, Bremen 1982 .

[7] D. Saupe, Global bifurcation of periodic solutions to some autonomous differential delay equations, Report Nr. 71, Forschungsschwerpunkt "Dynamische Systeme", Universität Bremen, Bremen 1982.