A Guided Tour of the Fractal Image Compression Literature

Dietmar Saupe, Raouf Hamzaoui
Universität Freiburg[1]
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Abstract. Since the conception of fractal image compression by Michael F. Barnsley around 1987 the research literature on this topic has experienced a rapid growth, which is hard to keep track of. This paper contains a brief description of the major advances in the field. The corresponding relevant papers are presented in the form of the original (slightly edited) abstracts. Also included is a comprehensive bibliography, the largest published on this topic to this date.

1 Introduction

While JPEG is becoming the industry standard for image compression technology there is ongoing research in alternative methods. Currently there are at least two exciting new developments: wavelet based methods and fractal image compression. This article is intended to provide the reader with an overview and a resource of the research on the latter. We attempt to put the work into a historical perspective and aim at providing the most comprehensive and up-to-date list of references in the field, which has truly been considerable in the number of publications as shown in the following table.

Year	1987	1988	1989	1990	1991	1992	1993	(1994)
Publications	1	7	7	9	17	23	31	(45)

The organization of the rest of this article proceeds as follows. In the next section we present a brief mathematical framework of fractal image compression. The third section provides an overview of the major advances in the research of fractal image compression starting from the visionary conception of Barnsley in 1987 and the ground-breaking work of Jacquin in his 1989 PhD thesis. The fourth section is a short list of available fractal compression software. In the fifth section we present a collection of original abstracts of what we think are the papers that contained important progress or were most influential. The sixth section contains two tables giving a quick survey of the material included in the

[1]Address: Institut für Informatik, Universität Freiburg, Rheinstr. 10–12, 79104 Freiburg, Germany. Email: {saupe|hamzaoui}@informatik.uni-freiburg.de. A previous version of this paper is included in the ACM SIGGRAPH'94 Course Notes, New Directions for Fractal Modelling in Computer Graphics, J. Hart (ed.). A short version of this paper will appear in Computer Graphics.

references. Finally we conclude with the bibliography which could be regarded as the main contribution of this article.

Since we explain the mathematical fundamentals involved in a fractal image compression scheme only at some coarse level of detail, the reader will profit the most from this paper when already familiar with the basic concepts. If necessary, such knowledge can be attained by reading introductory texts or reviews, for example, [BaHu92, Fish92a, Fish94a, Jacq93].

We believe that here we are presenting a fairly full picture of the literature. Of course, in spite of our efforts, some pieces containing relevant work may have skipped our attention and, thus, may have been unduly and unintentionally ignored here.[2]

2 The mathematical principle behind fractal image compression

As a model for the space of monochrome images we choose a space E of bounded continuous functions $f : X \to G$ for the simplicity of its mathematical description. The set X taken for example as the unit square represents the set of the spatial coordinates of the image while the set G taken as the interval $[0,1]$ represents the set of intensity values of the image. However, for practical applications suitable for computer processing one can prefer a discrete framework in which a spatially digitized image is modeled as a point of a finite dimensional space. Thus, a discrete grey-tone image of size $n \times m$ pixels is thought of as a point in $\mathbb{R}^{n \times m}$. After a distance d is constructed such that (E,d) is a complete metric space, the fractal (or attractor) coding of the image f is seen as the optimization problem:

> *Find a contractive operator T on (E,d) whose fixed point $g = Tg$ is the best possible approximation of f* (the contraction mapping principle ensures that a fixed point $g = Tg$ exists and is unique).

This optimization problem will be approached by means of the collage theorem [Barn88b]:

Collage Theorem. *Let T be a contraction on the complete metric space (E,d) with contractivity factor s and fixed point g. Let $f \in E$. Then $d(f,g) \leq \frac{1}{1-s} d(f,Tf)$.*[3]

Thus, by minimizing the distance between f and Tf (the collage of the image), we hope to minimize the distance between the fixed point g and the given image f. Of course, if the value of s is close to 1, nothing ensures that this method provides a good approximation. Yet this was the original idea of Barnsley and most of the fractal based algorithms rely on the same approach. The fractal compression scheme can be viewed as two consecutive steps.

1 The encoding process (see figure 1):

It consists of the construction of the operator T which will be defined by a set $\{(R_k, D_k, u_k, v_k), 1 \leq k \leq N\}$. The sets R_k, called *ranges*, form a partitioning of X. The sets

[2] To prevent this in future versions of this bibliographic article, readers are encouraged to send comments, preprints, reprints and technical reports, all of which will be gratefully appreciated.

[3] The contraction factor $s < 1$ of T satisfies the estimate $d(Tf_1, Tf_2) \leq s \cdot d(f_1, f_2)$ for all images f_1, f_2.

Figure 1: Elements of the fractal code. Left: partitioning of the image region (a square) into ranges. Center: some of the corresponding domains. Right: the affine transformation v_k for the k-th domain-range pair. For each domain-range pair (D_k, R_k) there is an (invertible) geometric transformation $u_k : D_k \to R_k$. The function f evaluates image intensities. For a point $x \in R_k$ we compute its preimage $u_k^{-1}(x)$ in the corresponding domain D_k, look up the image intensity, $f(u_k^{-1}(x))$, and finally apply the affine transformation, v_k, obtaining $Tf(x) = v_k f u_k^{-1}(x)$ for $x \in R_k$.

Figure 2: Encoding and decoding for a one-dimensional example. Let $f : [0,1] \to [0,1]$ denote the 'image' to be encoded (left). We choose the same domain $[1/2, 1]$ for all ranges shown on the left. Note that the graph of f in R_1 as well as in R_3 is just a scaled down copy of the graph in the domain. In R_2 we can reproduce f by a vertical flip of the same copy, in R_4 we use the copy with an additional offset of 1/2. The code thus employs the geometric transformations $u_k(x) = x/2 + (k-2)/4$ and the affine transformations $v_1(f) = v_3(f) = f/2$, $v_2(f) = (1-f)/2$, and $v_4(f) = (1+f)/2$. When reconstructing the original from the code we may start with an arbitrary 'image', g, and repeatedly apply the image operator T. The first two iterations Tg and $T^2 g$ are shown for the choice $g(x) = x$.

D_k, called *domains*, are also subsets of X but may overlap. For each R_k, a D_k, a bijection $u_k : D_k \to R_k$ and a contraction $v_k : G \to G$ (this map adjusts the intensity values in the domain to those in the range) are chosen such that the distance $d(f_{/R_k}, v_k f_{/u_k^{-1} R_k})$ is as small as possible. This is simply realizing the condition $f \approx Tf$ *locally* by exploiting the redundancy contained in the image since we seek for each part of the image corresponding to a range a similar (under appropriate contractive transformations) part corresponding to a domain. Finally, the operator T is given by $Tf = \Sigma_{k=1}^N T_k f$ where $T_k f(x) = 1_{R_k}(x) v_k(f(t(x)))$ and $t(x) = \Sigma_{k=1}^N 1_{R_k}(x) u_k^{-1}(x)$.

2 The decoding process (see figure 2):

It consists of the computation of the fixed point. This is accomplished by iterating the operator T upon any initial image f_0. Since the operator T is contractive, the contraction mapping principle ensures the convergence of the sequence $\{T^n(f_0)\}$ to the fixed point.

This setup is called a *local iterated function system* or a *partitioned iterated function system* (PIFS).

3 Overview

Fractal image compression is based on the concepts and mathematical results of iterated function systems (IFS). The roots of this theory are at least 10–20 years old (see the work of Williams[4] and Hutchinson[5]). Then, in the mid 1980's, IFS's became very popular. It was Barnsley and his coworkers at Georgia Institute of Technology who first noticed the potential of IFS for applications in computer graphics. Initially, around 1985, their research focussed on *modelling* natural shapes such as leaves and clouds,[6] but then Barnsley and Sloan advertised in popular science magazines the incredible power of IFS also for compressing color images at compression rates of over 10000 to 1, see, for example, [BaSl87, BaSl88, SciAm88]. They included a few decoded images supporting this claim. The algorithms that were used to generate these astonishing results consisted of two phases.[7] First, an image had to be segmented into parts, that were as self-similar as possible. Then each part was coded as an IFS with probabilities. The key for this was the collage theorem providing a criterion for the choice of the transformations in the IFS code thereby optimizing the overall result. For the decoding, the "chaos game" then produced a large number of points, the histogram of which serving as the approximation of the corresponding part of the image. Finally, the decoded parts had to be reassembled to produce the complete decoded IFS representation. While the decoding could proceed automatically, the encoding required human interaction, at least in the segmentation of the image. Barnsley and Sloan continued their work from within their newly formed company, Iterated Systems, Inc., devoted to applications of iterated function systems, especially fractal image compression. They were granted two patents [BaSl90, BaSl91] and since then the company has offered commercial image compression software and hardware. After *Fractals Everywhere* [Barn88b], Barnsley and Hurd [BaHu92] have come out with a second book, which is dedicated to fractal image compression.

Several researchers have taken up the challenge to design an automated algorithm to solve the inverse (i.e., the encoding) problem using the basic IFS method and its generalizations (recurrent iterated function systems, RIFS). Vrscay and Forte have studied the so-called moment method [Vrsc91a, Vrsc91b, FoVr94a, FoVr94b], Bedford, Dekking and Keane

[4] Williams, R. F., *Compositions of contractions,* Bol. Soc. Brasil. Mat. 2 (1971) 55–59.

[5] Hutchinson, J., *Fractals and self-similarity,* Indiana Univ. Math. J. 30 (1981) 713–747.

[6] S. Demko, L. Hodges, and B. Naylor, *Construction of fractal objects with iterated function systems,* Computer Graphics 19,3 (1985) 271–278.

[7] This is not the method presented in the previous section.

[BeDeKe92] have tried the simulating annealing method, studied the general IFS approach theoretically and came to the conclusion that there are considerable mathematical obstacles in approximating images in this way.

In 1989 Jacquin proposed the first fully automated algorithm for fractal image compression. It was based on affine transformations acting locally rather than globally. This new approach first appeared in his PhD thesis [Jacq89] and since then several papers [Jacq90a, Jacq90b, Jacq92] have popularized his scheme. A digital monochrome image is partitioned into nonoverlapping square pixel blocks (range blocks). Larger square pixel blocks (domain blocks) which may overlap are sorted into a set of categories (shade blocks, edge blocks and midrange blocks) following a classification, well-known in image processing. For each range block, a domain block of the same category is searched (for evident complexity reduction purposes) such that its image under a local strictly contractive affine mapping minimizes its distance to the original block in the root mean squares metric. Each affine mapping is composed of a *geometric* part which shrinks the domain block down to the size of a range block by pixel averaging, and a *massic* part that transforms the obtained block by shuffling (8 alternatives corresponding to the isometry group of the square), scaling with quantized parameters, and addition of a constant grey-tone block. These operations were called contrast scaling and luminance shift respectively. The union of the affine mappings, the Jacquin block operator, is shown to be contractive on the set of discrete images. The iteration of the block operator upon any initial image generates an approximation of the target image. This scheme is by many aspects related to vector quantization with which it shares the idea of using a codebook providing a library for the selection of the domain blocks. However, the codebook in fractal compression is only a "virtual" one since the domain blocks are not stored but taken from the image itself thus exploiting the redundancy of the information present in the image.

In a way, the thesis of Jacquin and his follow-up papers broke the ice for fractal image compression providing a starting point for further research and extensions in many possible directions. Some of the main subjects addressed so far are:

- the partitioning of the image into ranges: adaptive quadtrees, rectangular and triangular ranges,
- the encoding: choice of the domain pool, including several fixed basis blocks and even several image domain blocks for the code of a range, choice of the transformations defining the operator,
- classification methods for the complexity reduction of the encoding step: based on image values and intensity variance, clustering of domains, fast algorithms from computational geometry to solve nearest neighbor problems,
- the decoding: standard iteration versus fast hierarchical or direct numerical,
- coding of 1D or 3D data: time series, volume data, video frames.

These aspects were studied in order to obtain the best compromise in the three key issues of every image compression scheme, namely:

- image fidelity,
- compression ratio,

• time complexity of the encoder/decoder.

Most papers consider several of these aspects. Thus, rather than presenting the main ideas and advances in these topics one by one we proceed by discussing the results of the different research groups roughly (but not precisely) in chronological order.

In [FiJaBo92], Fisher, Jacobs and Boss introduced adaptive methods in the encoding. They used quadtree, rectangular, and triangular partitions of the range blocks to improve the image fidelity. They also pointed out the important fact that it is not necessary to impose strict contractivity conditions on the transformations of the code since the eventual contractivity [8] of their union is a sufficient condition to ensure the convergence of the iteration process in the decoding. Their classification scheme [Fish94a] is made with a clever design of a variable number of classes (4–12–72) taking into account not only intensity values but also intensity variance across a domain. In [JaFiBo92], they study the dependence of the performance of the encoding scheme on the quantization of the scale factor for the contrast scaling and the offset for the luminance shift, the number of domains used, the number of domain classes searched, the tolerance level employed to stop the adaptive algorithm, the maximum allowed value for the contractivity factors of the maps and the minimum range size in the quadtree subdivision. The effects of postprocessing the image by minimizing the discontinuities at the block boundaries are also considered.

Independently from Fisher et al, Bedford, Dekking and Keane [BeDeKe92] implemented a similar scheme based on quadtree partitions. Like Fisher et al they also noted that the search for the factors providing the scale factor and offset can be computed directly by solving a least squares problem (this approach appeared earlier in [OiLeRa91]). Finally, they introduced Rademacher functions to propose a criterion for the elimination of bad domains.

The work of Lundheim [Lund92, Lund94] presents a systematic analysis of the fractal encoding and decoding problem based on a discrete setting, i.e., emphasizing the finite-dimensional and discrete nature of digital signals. Besides a discussion of optimal collages in the usual least squares sense using affine operators, other norms are discussed (e.g., in the context of this work the Hutchinson metric is shown to be a weighted l_1 norm). Several new results are obtained in this approach. For example, eventual contractivity may be tested based on an efficient way of computing contraction factors which also lead to improved versions of the collage theorem. Furthermore, an interesting noniterative decoding method is presented which reportedly runs faster than the usual iterative one.

Using the mathematical framework described by Lundheim [Lund92] in which blocks of a discrete grey-tone image are seen as points of a finite dimensional inner product space, Lepsøy and Øien [Leps93, Oien93, LeOi94, OiLe94] generalized Jacquin's algorithm by letting the translation term be spanned by several basis vector blocks (see also [OiLeRa91] and [GhHu93]). By making all the decimated domain blocks orthogonal to the translation subspace basis vectors which were previously orthogonalized by a Gram-Schmidt procedure, it was shown that the l_2 optimization of the collage was computationally less expensive than an optimization without orthogonalization. But a more fundamental fact is obtained. The optimization can be done without constraining the size of the scaling coefficients as it is the

[8] A transformation t is eventually contractive if there exists a positive integer n such that t^n is contractive.

case in [Jacq92, BeDeKe92, FiJaBo92]! The orthogonalization operator will always make the decoding algorithm converge exactly in a finite number of iterations.

An adaptive technique for block classification is done by a *clustering* of the codebook (the set of shrunk, shuffled domain blocks). The codebook is subdivided into subsets by computing centers and grouping the codebook blocks around the centers. In the encoding, a range block is *compared* first to the centers and then to the blocks in the corresponding cluster. The criterion for comparing blocks relies on a similarity measure which is large when the blocks are parallel.

Fractal compression based on piecewise self-similarities has first been implemented by Jacquin for images, i.e., for digital signals in two dimensions. Of course, the same ideas are applicable for modelling one-dimensional signals. The group at the Department of Electrical Engineering at the Georgia Institute of Technology consisting of Hayes, Mazel and Vines has investigated this application in a number of papers. For example, in [MaHa92] the approach using linear fractal interpolation as well as the piecewise self-affine fractal model are discussed with algorithms that are adaptive in the choice of the sizes of the ranges and domains. Some previous work on 1D-coding is in [MaSl89, With89].

An interesting new variant of fractal image coding, developed by Vines in [Vine94], is given by an orthonormal basis approach which is a hybrid method combining principles of transform coding with those of fractal decoding. An image range is covered by a linear combination of fixed basis blocks and image basis domain blocks. The fixed basis blocks are determined a priori. For example, one can use three blocks giving all bivariate polynomials of degree one, or six blocks producing all such polynomials of degree two. (Such fixed basis blocks had already been introduced in [OiLeRa91].) If the scheme is designed for range blocks of, say, 8 by 8 pixels, then another set of blocks taken from the original image (down-filtered to size 8 by 8) are chosen to make up a total of 64 linearly independent blocks. Each range block is then approximated by a linear combination of only a few of the fixed basis blocks and the image basis blocks. To make this approach efficient, care must be taken that the set of chosen image blocks fits well to the set of all range blocks. Also domain blocks must be orthogonalized (or required to be almost orthogonal) to facilitate easy computation of the coefficients. See the reference [GhHu93]. The decoding must use the iteration procedure common to fractal image compression, where in each iteration the evaluation of the linear combination is computed for each range.

An original approach to fractal coding is described in [MoDu92a, MoDu92b]. The image is partitioned into nonoverlapping rectangular blocks. Each block is split into a finite number of tiles using an IFS. Then, each tile is coded by a least-squares approximation of the transformed block (under an affine mapping). Thus, the encoding is accomplished without searching by solving a set of linear equations whose coefficients are computed in linear time with the total number of pixels (see also [LiNoFo93] for a comparable technique in 3 dimensions). This method called the Bath fractal transform is generalized in [Monr93a, Monr93b, Monr93c] by including searching at different levels for which the cost/image fidelity trade-off is experimentally investigated. The results indicate that the fidelity gained by searching does not compensate the extra bits needed to specify the symmetries. It is suggested that the use of higher order contractive maps (instead of the affine ones) could be a better option

[MoWo94]. In [Dudb94], Dudbridge presented a similar coding method with a fast non-iterative decoding algorithm. Some promising results on fractal video compression are reported in [MoNi94, WiNiMo94].

In [BaMaKa93, BaMaKa94], Baharav, Malah, and Karnin proposed a fast decoding algorithm based on a hierarchical interpretation of the IFS-code. Essentially the methods prescribes the usual iteration for the decoding, however, with the modification, that the dimension of the underlying space (i.e., the size of the image that is being iterated) grows from one iteration to the next. Thus, initially, when the dimension is small, an iteration is very fast, while the full size image is used only in the last iteration. Of course, an interpolation procedure must be carried out between iterations in order to increase the dimension. Although the mathematics of this are described only for a special case, it is clear that the method can be successfully applied in practice when the strict conditions, which are only convenient in the derivation of the mathematical proof are not fulfilled. In [OBLMK94], with the collaboration of Øien and Lepsøy, they present a new collage theorem holding for a certain class of affine mappings called Affine Blockwise Averaging maps which operate on the space of discrete signals and are suitable for the orthogonalized version of Jacquin's operator introduced in the theses of Øien and Lepsøy [Oien93, Leps93]. The theorem provides a better bound on the distance between the original image and the attractor by considering in the estimate norms of collage errors at successively coarser resolutions. The improvement tested on real world images is reported to be vast.

4 Software

There exists commercial software and hardware offered by Barnsley's company, Iterated Systems, Inc. Information can be obtained from

Iterated Systems, Inc.
5550-A Peachtree Parkway
Norcross, GA 30092

Besides this commercial software we have found three sources for public domain programs:

- The code for Fisher's adaptive quadtree method can be obtained via ftp from legendre.ucsd.edu in pub/Research/Fisher. Among other things there is C code to encode and decode raw byte files of any size using a quadtree method, a manual explaining the use of the code, some sample encodings, and the SIGGRAPH '92 course notes on fractal image compression (reference [Fish92b], which is based on [Fish92a]).
- Another fractal compression program is available by ftp in vision.auc.dk: /pub/Limbo/Limbo*.tar.Z.
- The source code for the program published in the reference [Anso93] is in ftp.uu.net:/published/byte/93oct/fractal.exe. This code is for a TARGA video board.

These sources existed at the time of the preparation of this article. Of course, at some undefined time in the future they will have changed or even disappeared.

5 Selected Abstracts

The following abstracts are adapted from the original ones which were slightly edited to suit a uniform format.

[BJMRS88]
Harnessing Chaos for Image Synthesis
M. F. Barnsley, A. Jacquin, F. Malassenet, L. Reuter, A. D. Sloan
Computer Graphics 22,4 (1988) 131–140.

Chaotic dynamics can be used to model shapes and render textures in digital images. This paper addresses the problem of how to model geometrically shapes and textures of two dimensional images using iterated function systems. The successful solution to tis problem is demonstrated by the production and processing of synthetic images encoded from color photographs. The solution is achieved using two algorithms: (1) an interactive geometric modeling algorithm for finding iterated function system codes; and 2) a random iteration algorithm for computing the geometry and texture of images defined by iterated function system codes. Also, the underlying mathematical framework, where these two algorithms have their roots, is outlined. The algorithms are illustrated by showing how they can be used to produce images of clouds, mist and surf, seascapes and landscapes and even faces, all modeled from original photographs. The reasons for developing iterated function systems algorithms include their ability to produce complicated images and textures from small databases, and their potential for highly parallel implementation.

[BaSl88]
A Better Way to Compress Images
M. F. Barnsley and A. D. Sloan
BYTE, January 1988.

The natural world is filled with intricate detail. Consider the geometry on the back of your hand: the pores, the fine lines, and the color variations. A camera can capture that detail and, at your leisure, you can study the photo to see things you never noticed before. Can personal computers be made to carry out similar functions of image storage and analysis? If so, then image compression will certainly play a central role. The reason is that digitized images — images converted into bits for processing by a computer — demand large amounts of computer memory. For example, a high-detail grey-scale aerial photograph might be blown up to a 3 1/2-foot square and then resolved to 30 by 300 pixels per square inch with 8 significant bits per pixel. Digitization at this level requires 130 megabytes of computer memory — too much for personal computers to handle. For real-world images such as the aerial photo, current compression techniques can achieve ratios of between 2 to 1 and 10 to 1. By these methods, our photo would still require between 65 and 13 megabytes. In this article, we describe some of the main ideas behind a new method for image compression using fractals. The method has yielded compression ratios in excess of 10,000 to 1 (bringing our aerial photo down to a manageable 13,000 bytes).

[Jacq92]
Image Coding Based on a Fractal Theory of Iterated Contractive Image Transformations
A. E. Jacquin
IEEE Trans. Image Processing 1 (1992) 18–30.

In this paper, we propose an independent and novel approach to image coding, based on a fractal theory of iterated transformations. The main characteristics of this approach are that i) it relies on the assumption that image redundancy can be efficiently exploited through *self-transformability* on a blockwise basis, and ii) it approximates an original image by a *fractal block coding*. The coding-decoding system is based on the construction, for an original image to encode, of a specific image transformation — a fractal code — which, when iterated on any initial image, produces a sequence of images which converges to a fractal approximation of the original. We show how to design such a system for the coding of monochrome digital images at rates in the range of 0.5–1.0 b/pixel. Our fractal block coder has performance comparable to state-of-the-art vector quantizers, with which it shares some aspects. Extremely promising coding results are obtained.

[BeDeKe92]
Fractal Image Coding Techniques and Contraction Operators
T. Bedford, F. M. Dekking and M. S. Keane
Nieuw Arch. Wisk. (4) 10,3 (1992) 185–218.

Michael Barnsley and his coworkers have recently popularized the idea of using fractals to achieve compression of the data in an image (for example, a TV picture or a satellite photograph). The intuitive idea is very appealing: Since many images contain complex fractal-like objects, an efficient way of encoding the image will try to describe these objects as fractals rather than approximating them in "smooth" ways. Barnsley and Sloan [BaSl88] wrote an article for a popular computing journal in which a method for fractal coding was loosely described, a number of "coded" pictures were shown, and the astonishing claim was made that compression rates of 1:10 000 are possible using their techniques. Since then Barnsley and Sloan have been granted a US patent [BaSl90] on "Methods and apparatus for image compression by iterated function system". In this paper we shall discuss some of the problems involved with any implementation of Barnsley's coding methods. We also discuss an alternative fractal coding technique due to Jacquin [Jacq89, Jacq92], which was developed as part of his PhD thesis under Barnsley at Georgia Institute of Technology.

[FiJaBo92]
Fractal Image Compression Using Iterated Transforms
Y. Fisher, E. W. Jacobs, and R. D. Boss
in: *Image and Text Compression,* J. A. Storer (ed.), Kluwer Acad. Publ., Boston, 1992.

This article presents background, theory, and specific implementation notes for an image compression scheme based on fractal transforms. Results from various implementations are presented and compared to standard image compression techniques.

[**JaFiBo92**]
Image Compression: A Study of the Iterated Transform Method
E. W. Jacobs, Y. Fisher, and R. D. Boss
Signal Processing 29 (1992) 251–263.

This paper presents results from an image compression scheme based on iterated transforms. Results are examined as a function of several encoding parameters including maximum allowed scale factor, number of domains, resolution of scale offset values, minimum range size, and target fidelity. The performance of the algorithm, evaluated by means of fidelity versus the amount of compression, is compared with an adaptive discrete cosine transform image compression method over a wide range of compressions.

[**MoDu92a**]
Fractal Approximation of Image Blocks
D. M. Monro and F. Dudbridge
Proc. ICASSP 3 (1992) 485–488.

A method for block coding of images is presented, based on a least squares fractal approximation by a Self Affine System (SAS). The computational cost of the approximation is linear in the number of pixels in the image. The approximation to a rectangularly tiled block involves evaluating various low order moments over the block, and solving a system of four linear equations for each tile. The method is applied to a standard test image, and the effects of various optimizations are shown. A quantitative comparison with the Adaptive Discrete Cosine Transform at 8:1 compression is made. The fidelity of the fractal method shows promise, and its greater speed and simplicity compared to other fractal transforms suggest immediate applications such as interactive browsing of remote image archives or image representation in multimedia systems.

[**MaHa92**]
Using Iterated Function Systems to Model Discrete Sequences
D. S. Mazel and M. H. Hayes
IEEE Transactions on Signal Processing 40,7 (1992) 1724–1734.

In this paper, two iterated function system models are explored for the representation of single-valued discrete-time sequences: the self-affine fractal model and the piecewise self-affine fractal model. We present algorithms, one of which is suitable for a multiprocessor implementation, for identification of the parameters of each model. Applications of these models to a variety of data types are given where signal-to-noise ratios are presented, quantization effects of the model parameters are investigated, and compression ratios are computed.

[**Lund92**]
Fractal Signal Modellings for Source Coding
L. M. Lundheim
PhD Thesis, The Norwegian Institute of Technology, Trondheim, September 1992.

The present thesis deals with analysis and synthesis methods for model-based source coders. The models in questions are *fractal*, meaning that they assume some kind of *self similarity* as an inherent property of the messages to be coded, and that simple, usually iterative algorithms exist for synthesizing the decoded messages. A new *discrete domain* setting is presented for formulating and solving the problem, leading to simpler mathematical treatment than earlier approaches using measures or functions on continuous domains. It is demonstrated that the new framework includes sampled versions of both Weierstrass functions and Barnsley's fractal interpolation functions. Both analysis and synthesis is treated using finite dimensional, normed vector spaces and operators on these. Emphasis is put on finding *optimal collages in the least squares sense using affine operators,* but use of other norms and other kinds of non-linear operators is also discussed. In particular, it is shown how the Hutchinson metric may be interpreted as a weighted l_1 norm. Applications of the methods are demonstrated for audio signals and images. For a particular class of affine operators it is shown how eventual contractivity may be controlled by regarding the eigenvalues of a certain matrix. An efficient method for computing these eigenvalues is derived. For the same class of operators an efficient non-iterative synthesis method is presented, as well as a strengthening of the collage theorems.

[Oien93]
Optimal Attractor Image Coding with Fast Decoder Convergence
G. E. Øien
PhD Thesis, The Norwegian Institute of Technology, Trondheim, April 1993.

The thesis deals with a novel method for modelling and lossy compression of signals, with applications to real-world images. The attractor coding method is based on the assumption that real-world images possess some form of *self-similarity*, and thus can be modelled as such attractors. A modelling approach suited for spatially discrete signals is developed. Linear algebra and projection methods in linear vector spaces are used to optimize and generalize the model within certain chosen constraints. The problem of decoder convergence is treated in detail. A modification of previously known attractor coding algorithms is introduced, which makes it possible to determine, control, and significantly reduce the number of decoder iterations needed. This is done without sacrifices in the attainable image quality, and provides a decoding algorithm of very low computational complexity compared to that of almost any other source coding method. The modification also makes it easier to discuss the effect of some basic parameter choices, and to make analogies to other compression methods. The complexity of the attractor encoding algorithm, which basically is a large search problem, is analyzed and found to be prohibitively high. Modifications and simplifications of the basic algorithm are suggested, leading to a great reduction in encoder complexity. This is obtained by combining image block classification and dimensionality reduction to simplify the search problem, and leads to only small sacrifices in image quality. Parameter quantization is also discussed. The modification that leads to fast decoder convergence is also found to have positive consequences with respect to the ease of quantization. We consider scalar pdf-optimized quantization and uniform quantization of the real coefficients which are a part of the image code. We also present a bit allocation algorithm, which may be suitable for uniform quantization of the coefficients in the most general signal model considered in the thesis.

[Leps93]

Attractor Image Compression: Fast Algorithms and Comparisons to Related Techniques

S. Lepsøy

PhD Thesis, The Norwegian Institute of Technology, Trondheim, June 1993.

This dissertation deals with a family of techniques of digital still images, called attractor (or fractal) image compression. The aim is twofold; to develop algorithms for fast encoding and decoding, and to relate attractor image compression to other techniques experimentally and theoretically. Fast encoding is attained by reducing the complexity of a search in a large codebook. The codebook is given a tree-structure which is adapted to the original image (the image to be encoded). The adaptive structuring and subsequent encoding is approximately 8 times faster than encoding with an exhaustive search. The cost of this complexity reduction is a small degradation of the quality in the decoded images. A sufficient condition for attaining a fast decoder is formulated leading to an analytically and computationally simple expression for the decoded image.

[GhHu93]

A Fractal-Based Image Block-Coding Algorithm

M. Gharavi-Alkhansari and T. S. Huang

Proc. ICASSP 5 (1993) 345–348.

This paper presents a new block-coding algorithm for grey-scale images based on a fractal approximation. Image blocks are approximated by a linear combination of a set of almost orthogonal basis blocks made up of, (i) a set of simple image-independent blocks, and (ii) a set of image-*dependent* blocks generated from transformed blocks of bigger size in the same image. To find the approximation for each block, the block is projected onto the space spanned by the set of basis blocks. For each block, the largest coefficients plus the index of the corresponding basis blocks and the number of basis blocks used make the code. The number of basis vectors used for coding each block is the number of basis blocks needed to approximate the block within a given error limit. We also present the results of a study on the effect of variations on Jacquin's fractal-based image coding algorithm.

[BaMaKa93]

Hierarchical Interpretation of Fractal Image Coding and Its Applications to Fast Decoding

Z. Baharav, D. Malah and E. Karnin

in: *Intl. Conf. on Digital Signal Processing,* Cyprus, July 1993.

The basics of a block oriented fractal image coder are described. The output of the coder is an IFS code, which describes the image as a fixed-point of a contractive transformation. A new hierarchical interpretation of the IFS code, which relates different scales of the fixed-point to the code, is presented and proved. The proof is based on finding a function of a continuous variable, from which different scales of the original can be derived. Its application to a fast decoding algorithm is then described, leading typically to an order of magnitude reduction of computation time.

[Saup94a]
Breaking the Time Complexity of Fractal Image Compression
D. Saupe
Technical Report 53, Institut für Informatik, Universität Freiburg, 1994.

Fractal image compression allows fast decoding but suffers from long encoding times. This paper introduces a new twist for the encoding process. During encoding a large pool of image subsets, called domains, has to be searched repeatedly many times, which by far dominates all other computations in the encoding process. If the number of domains in the pool is N, then the time spent for each search is *linear* in N, $O(N)$. Previous attempts to reduce the computation times employ *classification schemes* for the domains based on image features such as edges or bright spots. Thus, in each search only domains from a particular class need to be examined. However, this approach reduces only the factor of proportionality in the $O(N)$ complexity. We suggest to replace the domain classification by a small set of real-valued keys for each domain. These keys are carefully constructed such that searching in the domain pool can be restricted to the *nearest neighbors* of a query point. Thus, we may substitute the sequential search in the domain pool (or in one of its classes) by multi-dimensional nearest neighbor searching. There are well known data structures and algorithms for this task which operate in *logarithmic* time, $O(\log N)$, a definite advantage over the $O(N)$ complexity of the sequential search. These time savings may provide a considerable acceleration of the encoding and, moreover, an enlargement of the domain pool potentially yielding improved image fidelity. A condensed version of this report appears in [Saup94b].

[BaHu92]
Fractal Image Compression
M. F. Barnsley and L. P. Hurd
AK Peters, Wellesley, 1992.

Fractal image compression involves three basic mathematical modelling problems. The required models are (1) a mathematical model for real world images, (2) a model for approximating model images by means of resolution independent image approximants, which must be described by finite data strings, and (3) a computationally tractable model for the sources of the data string, to enable the application of information theory to provide efficient representation of the image approximants. This book describes various models for each of these steps. It also includes C source code for applications of a number of these models. These applications clarify understanding of how the models work in a digital environment.
The chapters of this book and their contents in keywords are:

- Formulation of Mathematical Models for Real World Images: Scanning, Digitizing, Quantization, Color.
- Mathematical Foundations for Fractal Image Compression I: Affine transformations, topological properties of metric spaces and transformations, contraction mapping theorem.
- Fractal Image Compression I — IFS Fractals: Hausdorff space, iterated function systems, attractor, the collage theorem, measures and IFS's with probabilities, Dudbridge's fractal image compression method.
- Mathematical Foundations for Fractal Image Compression II: Information sources, Markov

sources, Kraft's theorem, entropy, Shannon-Fano codes, Huffman codes, arithmetic compression.

- Fractal Image Compression II — The Fractal Transform: Local iterated function systems, C source code.
- JPEG Image Compression: Discrete cosine transform, C code illustrating JPEG compression.

[Fish94a]
Fractal Image Compression — Theory and Applications to Digital Images,
Y. Fisher
Springer-Verlag, New York, 1994.

Here is a brief list of the book's highlights:

- An elementary introduction containing almost no mathematics.
- Rigorous description of all the relevant mathematics of the subject.
- Recent theoretical results on fast encoding and decoding methods, various schemes for encoding images using fractal concepts, and theoretical models for the encoding/decoding process.
- Working C code for a fractal encoding/decoding scheme capable of encoding images in a few seconds, decoding at arbitrary resolution, and achieving high compression ratios.
- Experimental results from various schemes showing their capability and forming the basis for a sophisticated implementation.
- A list of previously unresearched projects containing both new ideas and enhancements to the schemes discussed in the book.
- A comparison of the fractal schemes in the book with JPEG, commercial fractal software, and wavelet methods.

The titles and authors of the chapters are:

1. Introduction (Y. Fisher)
2. Mathematical Background (Y. Fisher)
3. Fractal Image Compression with Quadtrees (Y. Fisher)
4. Archetype Classification in an Iterated Transformation Image Compression Algorithm (R. D. Boss and E. W. Jacobs)
5. Hierarchical Interpretation of Fractal Image Coding and its Applications (Z. Baharav, D. Malah, and E. Karnin)
6. Fractal Encoding with HV Partitions (Y. Fisher and S. Menlove)
7. A Discrete Framework for Fractal Signal Modelling (L. Lundheim)
8. A Class of Fractal Image Coders with Fast Decoder Convergence (G. E. Øien and S. Lepsøy)
9. Fast Attractor Image Encoding by Adaptive Codebook Clustering (S. Lepsøy and G. E. Øien)
10. Orthogonal Basis IFS (G. Vines)
11. A Convergence Model (B. Bielefeld and Y. Fisher)
12. Least-squares Block Coding by Fractal Functions (F. Dudbridge)

13. Inference Algorithms for WFA and Image Compression (K. Culik II and J. Kari)
A. Sample Code (Y. Fisher)
B. Exercises (Y. Fisher)
C. Projects (Y. Fisher and D. Saupe)
E. Comparison of Results (Y. Fisher)

6 A rapid overview of the references

To provide a rapid overview of the papers contained in the bibliography we propose two tables. The first one (Table 1) is devoted to works using the basic IFS method (see page 4). The second table (Table 2) deals with papers that consider local iterated function systems. The bullet symbol indicates a topic that was stressed in the paper.

1D	One dimension
2D	Two dimensions
3D	Three dimensions (video)
P	Partitioning of the image
Dec	Decoding
FBB	Fixed basis blocks
Enc	Encoding
MM	Moment method
FI	Fractal interpolation
GA	Genetic algorithms
SA	Simulated annealing
RIFS	Recurrent iterated function systems

7 Conclusion

Although fractal image coding is a relatively new technique it has acquired a performance comparable with other methods such as JPEG or vector quantization. Furthermore, the field of research is far from being exhausted since there are many directions that have not yet been fully investigated (e.g., the use of non-affine transformations, the combination of fractal coding with other techniques and extensions to volume data and video frames). The main advantages of the fractal compression scheme are its ability to provide high compression ratios for a large class of images, the speed of its decoding process and its multi-resolution properties. However, to arrive at an optimal algorithm which can outperform traditional techniques more attention needs to be devoted to the encoding process which still suffers from long computation times.

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paper	1D	2D	3D	MM	FI	GA	SA	RIFS
BJMRS88		•						
Dudb94		•						
Holl91		•			•			
JaBoFi90		•						•
LiNoFo93			•					
MaHa89	•				•			
MaHa91	•				•			
MaHa92	•				•			
MoDu92a		•						
MoDu92b		•						
MoNi94			•					
MoWo94		•						
RaKa93b		•					•	
RiZa93		•		•				
VeRu94		•	•			•		
ViHa92b	•				•			
ViHa93b	•				•			
Vrsc91a	•	•		•		•		•
Vrsc91b	•	•		•		•		
WiNiMo94			•					
With89	•				•			

Table 1

paper	1D	2D	3D	P	Dec	FBB	Enc
BaMaKa93	•	•			•		
BaMaKa94	•	•			•		
BaVo94		•					•
BaVoNo93		•					•
Beau91	•	•	•				
BDBKS92	•	•					•
BeDeKe92	•	•					•
BoJa91		•					•
BoJa94		•					•
ChDaBe93		•		•			
DaBeCh93		•		•			
DaCh94		•		•			
Fi92a		•		•			
Fi92b		•		•			
FiJaBo91		•					
FiJaBo92		•		•			
FiLa92		•					
FiMe94		•		•	•		•
FiRoSh94			•				
FGHS94		•					•
GhHu93		•			•	•	•
JaBoFi90		•		•			•
JaFiBo92		•		•			•
LeOi94		•				•	•
Lund94	•	•					
MaHa92	•						
Monr93a		•		•			•
Monr93b		•		•			•
Monr93c		•		•			•
Monr93d		•		•			•
Nova93b		•		•			•
OBLMK94		•					•
OiLe94		•			•		
OiLeRa91		•				•	•
RaLe93		•			•		
Saup94a		•					•
Saup94b		•					•
SaBa94	•	•					
SiMa93	•						
Vine94		•				•	•
WoHsKu93		•		•			•

Table 2

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