A Review of the Fractal Image Compression Literature

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Since the conception of fractal image compression by Michael F. Barnsley around 1987, the research literature on this topic has experienced a rapid growth. Following is a brief description of the major advances in the field and the largest, comprehensive bibliography published on this topic to date.

While JPEG is becoming the industry standard for image compression technology, there is ongoing research in alternative methods. Currently there are at least two exciting new developments: wavelet based methods and fractal image compression. This article is intended to provide the reader with an overview and a resource of the research on the latter. We attempt to put the work into historical perspective and to provide the most comprehensive and up-to-date list of references in the field, truly a considerable number as shown in the following table.

Year	Publications	Year	Publications
1987	1	1991	17
1988	7	1992	23
1989	7	1993	31
1990	9	(1994)	(34)

The organization of this article is as follows. In the next section we present a brief mathematical framework of fractal image compression. The third section provides an overview of the major advances in the research of fractal image compression starting from the visionary conception of Barnsley in 1987 and the ground-breaking work of Jacquin in his 1989 Ph.D. thesis. The fourth section contains two tables giving a quick survey of the material included in the references. Finally we conclude with the bibliography which could be regarded as the main contribution of this article.

Since we explain the mathematical fundamentals involved in a fractal image compression scheme only at some coarse level of detail, the reader will profit the most from this paper when already familiar with the basic concepts. If necessary, such knowledge can be attained by reading introductory texts or reviews, for example, [BaHu92, Fish92a, Fish94a, Jacq93].

We believe that here we are presenting a fairly full picture of the literature. Of course, in spite of our efforts, some pieces containing relevant work may have skipped our attention and, thus, may have been unduly and unintentionally ignored here.

The mathematical principle behind fractal image compression

As a model for the space of monochrome images we choose a space E of bounded continuous functions $f: X \to G$ for the simplicity of its mathematical description. The set X taken for example as the unit square represents the set of the spatial coordinates of the image while the set G taken as the interval [0,1] represents the set of intensity values of the image. However, for practical applications suitable for computer processing one can prefer a discrete framework in which a spatially digitized image is modeled as a point of a finite dimensional space. Thus, a discrete grey-tone image of size $n \times m$ pixels is thought of as a point in $R^{n \times m}$.

After a distance d is constructed such that (E,d) is a complete metric space, the fractal (or attractor) coding of the image f is seen as the optimization problem:

Find a contractive operator T on (E,d) whose fixed point g=Tg is the best possible approximation of f (the contraction mapping principle ensures that a fixed point g=Tg exists and is unique).

This optimization problem will be approached by means of the collage theorem [Barn88b]:

Collage theorem

Let T be a contraction on the complete (E,d) metric space with contractivity factor s and fixed point g.
Let $f \in E$. Then

$$d(f,g) \le \frac{1}{1-s} d(f,Tf).$$

Thus, by minimizing the distance between f and Tf (the collage of the image), we hope to minimize the distance between the fixed point g and the given image f. Of course, if the value of s is close to 1, nothing ensures that this method provides a good approximation. Yet this was the original idea of Barnsley and most of the fractal based algorithms rely on the same approach. The fractal compression scheme can be viewed as two consecutive steps.

The encoding process (see figure 1)

It consists of the construction of the operator T which will be defined by a set. The sets R_R, called *ranges*, form a partitioning of X. The sets D_R called *domains*, are also subsets of X but may overlap. For each R_R, a D_R, a bijection $U_R: D_R \to R_R$ and a contraction $V_R: G \to G$ (this map adjusts the intensity values in the domain to those in the range) are chosen such that the distance $d(f_{/R_k}, v_k f_{/u_k^{-1} R_k})$ is as small as possible. This is simply realizing the condition $f=Tf$ *locally* by exploiting the redundancy contained in the image since we seek for each part of the image corresponding to a range a similar (under appropriate contractive transformations) part corresponding to a domain. Finally, the operator T is given by where $Tf = \sum_{k=1}^{N} T_k f$ and

$$t(x) = \sum_{k=1}^{N} 1_{R_k}(x) u_k^{-1}(x).$$

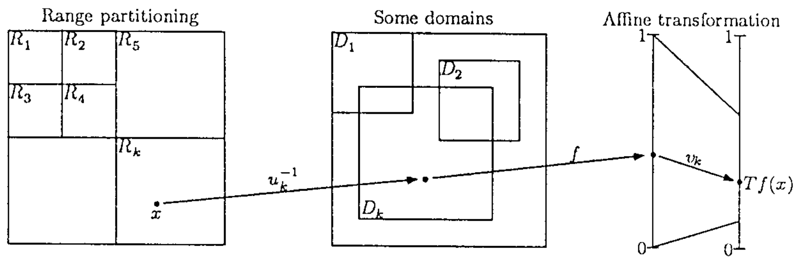

Figure 1. Elements of the fractal code. Left: Partitioning of the image region (a square) into ranges. Center: some of the corresponding domains. Right: the affine transformation V_k for the k-th domain-range pair. For each domain-range pair $D_k R_k$ there is an (invertible) geometric transformation $u_k: D_k \rightarrow R_k$. The function f evaluates image intensities. For a point $x \in R_k$ we compute its preimage u_k-$1(x)$ in the corresponding domain D, look up the image intensity, $f(u_k^{-1}(x))$and finally apply the affine transformation, V_k, obtaining $Tf(x) = v_k f u_k^{-1}(x)$ for $x \in R_k$.

The decoding process (see figure 2)

The decoding process consists of the computation of the fixed point. This is accomplished by iterating the operator T upon any initial image f_0. Since the operator T is contractive, the contraction mapping principle ensures the convergence of the sequence $\{T_n(f_0)\}$ to the fixed point.

Overview of fractal image compression research

Fractal image compression is based on the concepts and mathematical results of iterated function systems (IFS). The roots of this theory are at least 10-20 years old (see the work of Williams[2] and Hutchinson[3]). Then, in the mid 1980's, IFS's became very popular. It was Barnsley and his coworkers at Georgia Institute of Technology who first noticed the potential of IFS for applications in computer graphics. Initially, around 1985, their research focused on *modeling* natural shapes such as leaves and clouds[4] but then Barnsley and Sloan advertised in popular science magazines the incredible power of IFS for compressing color images at rates of over 10,000 to 1. [BaSl87, BaSl88, SciAm88]. They included a few decoded images supporting this claim.

The algorithms that were used to generate these astonishing results consisted of two phases.[5] First, an image had to be segmented into parts, that were as self-similar as possible. Then each part was coded as an IFS with probabilities. The key for this was the collage theorem providing a criterion for the choice of the transformations in the IFS code thereby optimizing the overall result. For the decoding, the "chaos game" then produced a large number of points, the histogram of which serving as the approximation of the corresponding part of the image. Finally, the decoded parts had to be reassembled to produce the complete decoded IFS representation. While the decoding could proceed automatically, the encoding required human interaction, at least in the segmentation of the image.

Barnsley and Sloan continued their work from within their newly formed company, Iterated Systems, Inc., devoted to applications of iterated function systems, especially fractal image compression. They were granted two patents [BaSl90, BaSl91] and since then the company has offered commercial image compression software and hardware. After *Fractals Everywhere* [Barn88b], Barnsley and Hurd [BaHu92] came out with a second book, which is dedicated to fractal image compression.

Several researchers have taken up the challenge to design an automated algorithm to solve the inverse (i.e., the encoding) problem using the basic IFS method and its generalizations (recurrent iterated function systems, RIFS). Vrscay and Forte have studied the so-called moment method [Vrsc91a,

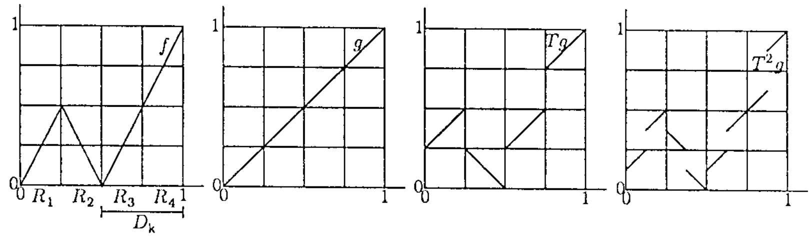

Figure 2. Encoding and decoding for a one-dimensional example. Let $f:[0,1] \rightarrow [0,1]$ denote the 'image' to be encoded (left). We choose the same domain [1/2,1] for all ranges shown on the left. Note that the graph of f in R_1 as well as in R_3 is just a scaled down copy of the graph in the domain. In R_2we can reproduce f by a vertical flip of the same copy, in R_4 we use the copy with an additional offset of 1/2 . The code thus employs the geometric transformations $u_k(x) = x/2 + (k-2)/4$ and the affine transformations $v_1(f) = v_3(f) = f/2$, $v_2(f) = (1-f)/2$, and $v_4(f) = (1+f)/2$. When reconstructing the original from the code we may start with an arbitrary 'image', g, and repeatedly apply the image operator T. The first two iterations T_g and T^2_g are shown for the choice $g(x) = x$.

This setup is called a *local iterated function system* or a *partitioned iterated function system* (PIFS).

Vrsc91b, FoVr94a, FoVr94b]. Bedford, Dekking and Keane [BeDeKe92] have tried the simulating annealing method, studied the general IFS approach theoretically and reached the conclusion that there are considerable mathematical obstacles in approximating images in this way.

In 1989 Jacquin proposed the first fully automated algorithm for fractal image compression. It was based on affine transformations acting locally rather than globally. This new approach first appeared in his Ph.D. thesis [Jacq89] and since then several papers [Jacq90a,Jacq90b,Jacq92] have popularized his scheme. A digital monochrome image is partitioned into nonoverlapping square pixel blocks (range blocks). Larger square pixel blocks (domain blocks) which may overlap are sorted into a set of categories (shade blocks, edge blocks and midrange blocks) following a classification, well-known in image processing. For each range block, a domain block of the same category is searched (for evident complexity reduction purposes) such that its image under a local strictly contractive affine mapping minimizes its distance to the original block in the root mean squares metric. Each affine mapping is composed of a *geometric* part which shrinks the domain block down to the size of a range block by pixel averaging, and a *massic* part that transforms the obtained block by shuffling (8 alternatives corresponding to the isometry group of the square), scaling with quantized parameters and addition of a constant grey tone block.

These operations were called contrast scaling and luminance shift respectively. The union of the affine mappings, the Jacquin block operator, is shown to be contractive on the set of discrete images. The iteration of the block operator upon any initial image generates an approximation of the target image. This scheme is by many aspects related to vector quantization with which it shares the idea of using a codebook providing a library for the selection of the domain blocks. However, the codebook in fractal compression is only a "virtual" one since the domain blocks are not stored but taken from the image itself, thus exploiting the redundancy of the information present in the image.

In a way, the thesis of Jacquin and his follow-up papers broke the ice for fractal image compression, providing a starting point for further research and extensions in many possible directions. Some of the main subjects addressed so far are:

- the partitioning of the image into ranges: adaptive quadtrees, rectangular and triangular ranges
- the encoding: choice of the domain pool, including several fixed basis blocks and even several image domain blocks for the code of a range and/or choice of the transformations defining the operator
- classification methods for the complexity reduction of the encoding step: based on image values and intensity variance, clustering of domains, fast algorithms from computational geometry to solve nearest neighbor problems
- the decoding: standard iteration versus fast hierarchical or direct numerical
- coding of 1D or 3D data: time series, volume data, video frames

These aspects were studied in order to obtain the best compromise in the three key issues of every image compression scheme, namely:

- image fidelity
- compression ratio
- time complexity of the encoder/decoder

Most papers consider several of these aspects. Thus, rather than presenting the main ideas and advances in these topics one by one we proceed by discussing the results of the different research groups roughly (but not precisely) in chronological order.

In [FiJaBo92], Fisher, Jacobs and Boss introduced adaptive methods in the encoding. They used quadtree, rectangular and triangular partitions of the range blocks to improve the image fidelity. They also pointed out the important fact that it is not necessary to impose strict contractivity conditions on the transformations of the code since the eventual contractivity[6] of their union is a sufficient condition to ensure the convergence of the iteration process in the decoding. Their classification scheme [Fish94a] is made with a clever design of a variable number of classes (4-12-72) taking into account not only intensity values but also intensity variance across a domain.

In [JaFiBo92], they study the dependence of the performance of the encoding scheme on the quantization of the scale factor for the contrast scaling and the offset for the luminance shift, the number of domains used, the number of domain classes searched, the tolerance level employed to stop the adaptive algorithm, the maximum allowed value for the contractivity factors of the maps and the minimum range size in the quadtree subdivision. The effects of postprocessing the image by minimizing the discontinuities at the block boundaries are also considered.

Independently from Fisher et al, Bedford, Dekking and Keane [BeDeKe92] implemented a similar scheme based on quadtree partitions. Like Fisher et al, they also noted that the search for the factors providing the scale factor and offset can be computed directly by solving a least squares problem (this approach appeared earlier in [OiLeRa91]). Finally, they introduced Rademacher functions to propose a criterion for the elimination of bad domains.

The work of Lundheim [Lund92, Lund94] presents a systematic analysis of the fractal encoding and decoding problem based on a discrete setting, i.e., emphasizing the finite-dimensional and discrete nature of digital signals. Besides a discussion of optimal collages in the usual least squares sense using affine operators, other norms are discussed (e.g., in the context of this work the Hutchinson metric is shown to be a weighted l_1 norm). Several new results are obtained in this approach. For example, eventual contractivity may be tested based on an efficient way of computing contraction factors which also lead to improved versions of the collage theorem. Furthermore, an interesting noniterative decoding method is presented which reportedly runs faster than the usual iterative one.

Using the mathematical framework described by Lundheim [Lund92] in which blocks of a discrete grey-tone image are seen as points of a finite dimensional inner product space, Lepsøy and Øien [Leps93, Oien93, LeOi94, OiLe94] generalized Jacquin's algorithm by letting the translation term be spanned by several basis vector blocks (see also [OiLeRa91] and [GhHu93]). By making all the decimated domain blocks orthogonal to the translation subspace basis vectors which were previously orthogonalized by a Gram-Schmidt procedure, it was shown that the l_2 optimization of the collage was computationally less expensive than an optimization without orthogonalization.

But a more fundamental fact is obtained. The optimization can be done without constraining the size of the scaling coefficients as it is the case in [Jacq92, BeDeKe92, FiJaBo92]! The orthogonalization operator will always make the decoding algorithm converge exactly in a finite number of iterations. An adaptive technique for block classification is done by a *clustering* of the codebook (the set of shrunk, shuffled domain blocks). The codebook is subdivided into subsets by computing centers and grouping the codebook blocks around the centers. In the encoding, a range block is *compared* first to the centers and then to the blocks in the corresponding cluster. The criterion for comparing blocks relies on a similarity measure which is large when the blocks are parallel.

Fractal compression based on piecewise self-similarities has first been implemented by Jacquin for images, i.e., for digital signals in two dimensions. Of course, the same ideas are applicable for modeling one-dimensional signals. The group at the Department of Electrical Engineering at the Georgia Institute of Technology consisting of Hayes, Mazel and Vines has investigated this application in a number of papers. For example, in [MaHa92] the approach using linear fractal interpolation as well as the piecewise self-affine fractal model are discussed with algorithms that are adaptive in the choice of the sizes of the ranges and domains. Some previous work on 1D-coding is in [MaSl89, With89].

An interesting new variant of fractal image coding, developed by Vines in [Vine94], is given by an orthonormal basis approach which is a hybrid method combining principles of transform coding with those of fractal decoding. An image range is covered by a linear combination of fixed basis blocks and image basis domain blocks. The fixed basis blocks are determined a priori.

For example, one can use three blocks giving all bivariate polynomials of degree one, or six blocks producing all such polynomials of degree two. (Such fixed basis blocks had already been introduced in [OiLeRa91].) If the scheme is designed for range blocks of, say, 8 by 8 pixels, then another set of blocks taken from the original image (down-filtered to size 8 by 8) are chosen to make up a total of 64 linearly independent blocks. Each range block is then approximated by a linear combination of only a few of the fixed basis blocks and the image basis blocks.

To make this approach efficient, care must be taken that the set of chosen image blocks fits well to the set of all range blocks. Also domain blocks must be orthogonalized (or required to be almost orthogonal) to facilitate easy computation of the coefficients. See the references [GhHu93]. The decoding must use the iteration procedure common to fractal image compression where in each iteration the evaluation of the linear combination must be computed for each range.

An original approach to fractal coding is described in [MoDu92a, MoDu92b]. The image is partitioned into nonoverlapping rectangular blocks. Each block is split into a finite number of tiles using an IFS. Then, each tile is coded by a least-squares approximation of the transformed block (under a contractive affine mapping). Thus, the encoding is accomplished without searching by solving a set of linear equations whose coefficients are computed in linear time with the total number of pixels (see also [LiNoFo93] for a comparable technique in 3 dimensions). This method, called the Bath fractal transform, is generalized in [Monr93a, Monr93b, Monr93c] by including searching at different levels for which the cost/image fidelity trade-off is experimentally investigated. The results indicate that the fidelity gained by searching does not compensate the extra bits needed to specify the symmetries. It is suggested that the use of higher order contractive maps (instead of the affine ones) could be a better option [MoWo94]. In [Dudb94], Dudbridge presented a similar coding technique with a fast non-iterative decoding algorithm. Some promising results on fractal video compression are reported in [MoNi94, WiNiMo94].

In [BaMaKa93, BaMaKa94], Baharav, Malah, and Karnin proposed a fast decoding algorithm based on a hierarchical interpretation of the IFS-code. Essentially the methods prescribe the usual iteration for the decoding. However, one modification is that the dimension of the underlying space (i.e., the size of the image that is being iterated) grows from one iteration to the next. Thus, initially, when the dimension is small, an iteration is very fast, while the full size image is used only in the last iteration. Of course, an interpolation procedure must be carried out between iterations in order to increase the dimension. Although the mathematics of this are described only for a special case, it is clear that the method can be successfully applied in practice when the strict conditions, which are only convenient in the derivation of the mathematical proof are not fulfilled.

In [OBLMK94], with the collaboration of Øien and Lepsøy, they present a new collage theorem holding for a certain class of affine mappings called Affine Blockwise Averaging maps which operate on the space of discrete signals and are suitable for the orthogonalized version of Jacquin's operator introduced in the theses of Øien and Lepsøy [Oien93, Leps93]. The theorem provides a better bound on the distance between the original image and the attractor by considering in the estimate norms of collage errors at successively coarser resolutions. The improvement tested on real world images is reported to be vast.

A rapid overview of the references

To provide a rapid overview of the papers contained in the bibliography, we propose two tables. The first is devoted to works using the basic IFS method. The second table deals with papers that consider local iterated function systems. The bullet symbol indicates a topic that was stressed in the paper.

1D .... One dimension
2D .... Two dimensions
3D .... Three dimensions (video)
P .... Partitioning of the image
Dec .... Decoding
FBB.......... Fixed basis blocks
Enc .... Encoding
MM .... Moment method
FI.............. Fractal interpolation
GA .... Genetic algorithms
SA .... Simulated annealing
RIFS......... Recurrent iterated function systems

Key

paper	1D	2D	3D	MM	FI	GA	SA	RIFS
BJMRS88		•						
Dudb94		•						
Holl91		•			•			
JaBoFi90		•						•
LiNoFo93			•					
MaHa89	•				•			
MaHa91	•				•			
MaHa92	•				•			
MoDu92a		•						
MoDu92b		•						
MoNi94			•					
MoWo94		•						
RaKa93b		•					•	
RiZa93		•		•				
VeRu94		•	•			•		
ViHa92b	•				•			
ViHa93b	•				•			
Vrsc91a	•	•		•		•		•
Vrsc91b	•	•		•		•		
WiNiMo94			•					
With89	•				•			

Table 1: Basic iterated function systems

paper	1D	2D	3D	P	Dec	FBB	Enc
BaMaKa93	•	•			•		
BaMaKa94	•	•			•		
BaVo94		•					•
BaVoNo93		•					•
Beau91	•	•	•				
BDBKS92	•	•					•
BeDeKe92	•	•					•
BoJa91		•					•
BoJa94		•					•
ChDaBe93		•		•			
DaBeCh93		•		•			
DaCh94		•		•			
Fi92a		•		•			
Fi92b		•		•			
FiJaBo91		•					
FiJaBo92		•		•			
FiLa92		•					
FiMe94		•		•	•		•
FiRoSh94			•				
FGHS94		•					•
GhHu93		•			•	•	•
JaBoFi90		•		•			•
JaFiBo92		•		•			•
LeOi94		•				•	•
Lund94	•	•					
MaHa92	•						
Monr93a		•		•			•
Monr93b		•		•			•
Monr93c		•		•			•
Monr93d		•		•			•
Nova93b		•		•			•
OBLMK94		•					•
OiLe94		•			•		
OiLeRa91		•				•	•
RaLe93		•			•		
Saup94a		•					•
Saup94b		•					•
SaBa94	•	•					
SiMa93	•						
Vine94		•				•	•
WoHsKu93		•		•			•

Table 2: Local iterated function systems

Conclusion

Although fractal image coding is a relatively new technique, it has acquired a performance comparable with other methods such as JPEG or vector quantization. Furthermore, the field of research is far from being exhausted since there are many directions that have not yet been fully investigated (e.g., the use of non-affine transformations, the combination of fractal coding with other techniques and extensions to volume data and video frames). The main advantages of the fractal compression scheme are its ability to provide high compression ratios for a large class of images, the speed of its decoding process and its multi-resolution properties. However, to arrive at an optimal algorithm which can outperform traditional techniques, more attention needs to be devoted to the encoding process which still suffers from long computation times.

Acknowledgments

We thank Zachi Baharav, Yuval Fisher, John Hart, Donald Monro and Geir Øien for stimulating discussions and for providing us with material. Kai Uwe Barthel, Tim Bedford, Franck Davoine, Mohammad Gharavi-Alkhansari, Bill Jacobs, Skjalg Lepsøy, Lars Lundheim, Mirek Novak, Isaac Rudomin and Edward Vrscay have also helped us by sending relevant papers.

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Footnotes

1 The contraction factor $s < 1$ of T satisfies the estimate $d(Tf1,Tf2) < s.d(f1,f2)$ for all images $f1,f2$.

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5 This is not the method presented in the previous section.

6 A transformation t is eventually contractive if there exists a positive integer n such that tn is contractive.

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