ON THE CONVERGENCE OF EULER-LIKE METHOD FOR THE SIMULTANEOUS INCLUSION OF POLYNOMIAL ZEROS

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Abstract–In this paper we give the convergence analysis of the Euler-like iterative method for the simultaneous inclusion of all simple real or complex zeros of a polynomial. The established initial conditions provide the safe convergence of the considered method and the fourth order of convergence. These conditions are computationally verifiable, which is of practical importance. A procedure for the choice of initial inclusion disks is also given.

Keywords: Polynomial zeros, simultaneous method, inclusion of zeros, convergence

1. INTRODUCTION

Iterative methods for solving polynomial equations, realized in complex interval arithmetic, produce resulting complex intervals (disks or rectangles) containing the zeros of a given polynomial. In this manner a control of rounding errors and information about upper error bounds of complex approximations to the zeros are provided (see [1] for more details). Moreover, in some practical problems of applied and industrial mathematics, algebraic polynomials with uncertain coefficients can appear. This kind of problems is effectively solved applying interval methods (c.f. [2], [3]).

The Euler-like method for the simultaneous inclusion of the zeros of a polynomial was derived and tested on numerical examples in the recent paper [4], but without details on the convergence rate. The aim of this paper is to present a detailed analysis of the convergence order, initial conditions for the convergence and the choice of initial disks which provide the guaranteed convergence of the Euler-like method. The stated conditions for the safe convergence are computationally verifiable, which is of practical importance.

The development and convergence analysis of the proposed algorithms need the basic properties of the so-called circular complex arithmetic, introduced by Gargantini and Henrici [5]. A circular closed region (disk) $Z := \{z : |z-c| \le r\}$ with center $c := \operatorname{mid} Z$ and radius $r := \operatorname{rad} Z$ we will denote by the parametric notation $Z := \{c; r\}$. If $Z_k := \{c_k; r_k\}$ $(k = 1, 2)$, then

$$\begin{aligned}
&Z_1 \pm Z_2 = \{c_1 \pm c_2; r_1 + r_2\},\\
&Z_1 \cdot Z_2 = \{c_1 c_2; |c_1| r_2 + |c_2| r_1 + r_1 r_2\},\\
&Z^{-1} = \{c; r\}^{-1} = \frac{\{\bar{c}; r\}}{|c|^2 - r^2} \quad (|c| > r,\ i.e.\ 0 \notin Z),\\
&Z_1 : Z_2 = Z_1 \cdot Z_2^{-1} \quad (0 \notin Z_2),
\end{aligned} \tag{1}$$

where the bar denotes the complex conjugate. Beside the exact inversion Z^{-1} of a disk, we also use the so-called centered inversion Z^I defined by

$$Z^I = \{c; r\}^I := \left\{ \frac{1}{c}; \frac{r}{|c|(|c| - r)} \right\} \supseteq Z^{-1} \quad (0 \notin Z). \tag{2}$$

The square root of a disk $\{c;r\}$ in the centered form, where $c=|c|e^{i\varphi}$ and $|c|>r$, is defined as the union of two disks (see [6]):

$$\{c;r\}^{1/2}:=\Big\{\sqrt{|c|}e^{i\frac{\varphi}{2}};\sqrt{|c|}-\sqrt{|c|-r}\Big\}\bigcup\Big\{-\sqrt{|c|}e^{i\frac{\varphi}{2}};\sqrt{|c|}-\sqrt{|c|-r}\Big\}. \tag{3}$$

For the basic interval operations $+,-,\cdot,:$ the *inclusion property* is valid, that is,

$$Z_k\subseteq W_k\Rightarrow Z_1*Z_2\subseteq W_1*W_2\quad (k=1,2;*\in\{+,-,\cdot,:\}).$$

Moreover, if f is a rational function and F its *complex circular extension*, then

$$Z_k\subseteq W_k\ (k=1,\dots,q)\Rightarrow F(Z_1,\dots,Z_q)\subseteq F(W_1,\dots,W_q).$$

Particularly, we have

$$w_k\in W_k\ (k=1,\dots,q;\ w_k\in\mathbb{C})\Rightarrow f(w_1,\dots,w_q)\in F(W_1,\dots,W_q).$$

In this paper we will use the following obvious properties:

$$\{c_1;r_1\}\subseteq\{c_2;r_2\}\iff |c_1-c_2|\le r_2-r_1, \tag{4}$$
$$\{c_1;r_1\}\cap\{c_2;r_2\}=\emptyset\iff |c_1-c_2|>r_1+r_2 \tag{5}$$

More details about circular arithmetic can be found in the book [7, Ch. 5]. Throughout this paper disks in the complex plane will be denoted by capital letters.

As in many papers on this subject, circular interval arithmetic is used in this paper too since it is more comfortable for calculations and manipulations. We note that the rectangular interval arithmetic can be also applied successfully; moreover, then the advantage of this type of interval arithmetic to incorporate rounding errors and reduce the interval result by using the intersection becomes evident.

2. EULER-LIKE INCLUSION METHOD

Let P be a monic polynomial with simple real or complex zeros $\zeta_1,\dots,\zeta_n$. Assume that we have found disjoint disks $Z_1,\dots,Z_n$ containing these zeros, that is, $\zeta_i\in Z_i$ for each $i\in I_n:=\{1,\dots,n\}$. Let $z_i=\operatorname{mid}Z_i\ (i\in I_n)$ be the center of the inclusion disk Z_i. We introduce the following abbreviations

$$w_i=\frac{P(z_i)}{\prod\limits_{\substack{j=1\\ j\ne i}}^{n}(z_i-z_j)},\quad g_{k,i}=\sum_{j\ne i}\frac{w_j}{(z_i-z_j)^k}\quad (k=1,2)$$

and sums

$$s_i=\sum_{j\ne i}\frac{w_j}{(z_i-z_j)(\zeta_i-z_j)},\quad S_i=\sum_{j\ne i}\frac{w_j}{(z_i-z_j)(Z_i-z_j)}\quad (i\in I_n).$$

The disk Z_i occurs more than once in the expression for S_i, but this does not influence the sharpness of a resulting disk since the addition of disks and the inverse of a disk are both exact operations in circular arithmetic.

The following fixed-point relation has been derived in [4]:

$$\zeta_i = z_i - \frac{2w_i}{1+g_{1,i} \pm \sqrt{(1+g_{1,i})^2+4w_i s_i}} \quad (i \in I_n). \tag{6}$$

Since $\zeta_i \in Z_i$ it follows $s_i \in S_i$ and, whence, using the inclusion isotonicity property,

$$\zeta_i \in \hat{Z}_i := z_i - \frac{2w_i}{1+g_{1,i} \pm \sqrt{(1+g_{1,i})^2+4w_i S_i}} \quad (i \in I_n). \tag{7}$$

If the disk Z_i is small enough so that the denominator in (7) does not contain the origin, then the set $\hat{Z}_i$ is again a circular disk which contains the zero ζ_i. In that case the sign "+" should be chosen in front of the square root in (7) (for details see [4]). According to this the following inclusion method has been stated in [4]:

ALGORITHM 1: *Let* $(Z_1,\dots,Z_n) =: \big(Z_1^{(0)},\dots,Z_n^{(0)}\big)$ *be initial disks such that* $\zeta_i \in Z_i$ $(i=1,\dots,n)$. *Writing* $z_i := \operatorname{mid} Z_i$ *and* $r_i := \operatorname{rad} Z_i$ *for the center and the radius of the disk* Z_i, *one step of the new Euler-like inclusion algorithm reads* $(Z_1,\dots,Z_n) \mapsto (\hat{Z}_1,\dots,\hat{Z}_n)$ *with*

$$\hat{Z}_i := z_i - \frac{2w_i}{1+g_{1,i} + \sqrt{(1+g_{1,i})^2+4w_i S_i}} \quad (i \in I_n). \tag{8}$$

The square root of a disk appearing in the denominator of (8) is calculated used (3). As shown in [4], the fixed point relation (6) is related in a certain extent to the classical Euler's third order method

$$\hat{z} = z - \frac{2f(z)}{f'(z) \pm \sqrt{f'(z)^2 - 2f(z)f''(z)}},$$

so that the inclusion method (8) has been named Euler-like method.

3. CONVERGENCE ANALYSIS

In this section we give initial conditions under which the Euler-like method (8) is convergent and show that its convergence rate is equal to four. Let us introduce the notation

$$r = \max_{1\le i\le n} r_i, \quad \rho = \min_{\substack{i,j\\ i\ne j}} \{|z_i - z_j| - r_j\}, \quad \alpha_n = \big(1 + 1/4(n-1)\big)^{n-1}.$$

In what follows we will always assume that $n \ge 3$. First we prove some necessary assertions.

LEMMA 1. *Assume that the inequality*

$$\rho > 4(n-1)r \tag{9}$$

holds. Then the following inequalities are valid:

(i) $\quad |w_i| < \alpha_n r < e^{1/4} r \approx 1.284\, r;$

(ii) $|g_{k,i}| < \dfrac{(n-1)\alpha_n r}{\rho^k} \quad (k = 1, 2, \dots);$

(iii) $|1 + g_{1,i}| > 1 - \dfrac{\alpha_n}{4};$

(iv) $\left|\dfrac{w_i g_{2,i}}{(1+g_{1,i})^2}\right| < \dfrac{1}{8}.$

PROOF OF (I): The sequence $(\alpha_n)_{n=2,3,\dots}$ is monotonically increasing with the upper bound $\lim_{n\to\infty} \alpha_n = e^{1/4}$. By virtue of this fact and (9) we have for each $j \in I_n$

$$|w_j| = \frac{|P(z_j)|}{\prod\limits_{\substack{k=1\\k\neq j}}^{n} |z_j - z_k|} = |z_j - \zeta_j| \prod_{\substack{k=1\\k\neq j}}^{n} \left|\frac{z_j - \zeta_k}{z_j - z_k}\right| < r_j \prod_{\substack{k=1\\k\neq j}}^{n} \frac{|z_j - z_k| + r_k}{|z_j - z_k|}$$

$$< r\Big(1 + \frac{r}{\rho}\Big)^{n-1} < r\Big(1 + \frac{1}{4(n-1)}\Big)^{n-1} = \alpha_n r < e^{1/4} r \approx 1.284r.$$

PROOF OF (II): Using (i) we find

$$|g_{k,i}| = \left|\sum_{j\neq i} \frac{w_j}{(z_i - z_j)^k}\right| \le \sum_{j\neq i} \frac{|w_j|}{|z_i - z_j|^k} < \frac{(n-1)\alpha_n r}{\rho^k}.$$

PROOF OF (III): By (9 and (ii) it follows

$$|1 + g_{1,i}| \ge 1 - |g_{1,i}| > 1 - \frac{(n-1)\alpha_n r}{\rho} > 1 - \frac{\alpha_n}{4}.$$

PROOF OF (IV): Let us define

$$\gamma_n := \frac{\alpha_n^2}{16(n-1)(1-\alpha_n/4)^2}, \; n > 1.$$

Using the estimate $\gamma_n \le \gamma_3 \approx 0.107 < 1/8$ for all $n \ge 3$, the inequality (9) and (iii) of Lemma 1, we obtain

$$\left|\frac{w_i g_{2,i}}{(1+g_{1,i})^2}\right| < \frac{\alpha_n^2 (n-1)(r^2/\rho^2)}{(1-\alpha_n/4)^2} < \frac{\alpha_n^2}{16(n-1)(1-\alpha_n/4)^2} = \gamma_n < \frac{1}{8}. \; \blacksquare$$

LEMMA 2. *If* (9) *holds, then the inclusion*

$$S_i = \sum_{j\neq i} \frac{w_j}{(z_i - z_j)(Z_i - z_j)} \subset \Big\{ g_{2,i}; \frac{r}{\rho^2} \sum_{j\neq i} \frac{|w_j|}{|z_i - z_j|} \Big\} \subset \Big\{ g_{2,i}; \frac{\alpha_n (n-1) r^2}{\rho^3} \Big\} \tag{10}$$

is valid.

PROOF: First, let us prove the inclusion

$$\frac{1}{Z_i - z_j} \subset \Big\{ \frac{1}{z_i - z_j}; \frac{r}{\rho^2} \Big\}. \tag{11}$$

Let us note that, since $|z_i - z_j| \geq \rho > 4(n-1)r > r_i$ (by (9)) it follows $z_j \notin Z_i$ so that the inverse disk $(Z_i - z_j)^{-1}$ exists. According to (4) the inclusion (11) will hold if the inequality

$$\left|\operatorname{mid}(Z_i - z_j)^{-1} - \frac{1}{z_i - z_j}\right| < \frac{r}{\rho^2} - \operatorname{rad}(Z_i - z_j)^{-1}$$

is valid. In regard to (1) the last inequality becomes

$$\left|\frac{\overline{z_i - z_j}}{|z_i - z_j|^2 - r_i^2} - \frac{1}{z_i - z_j}\right| < \frac{r}{\rho^2} - \frac{r_i}{|z_i - z_j|^2 - r_i^2},$$

or, after short rearrangement,

$$\frac{r_i(|z_i - z_j| + r_i)}{(|z_i - z_j| + r_i)(|z_i - z_j| - r_i)|z_i - z_j|} < \frac{r}{\rho^2},$$

which is true having in mind the definition of ρ and the assumption that the disks $Z_1, \ldots, Z_n$ are disjoint providing the inequality $|z_i - z_j| > r_i$.

Using (11) and circular arithmetic operations we find

$$\begin{aligned} S_i &= \sum_{j \neq i} \frac{w_j}{(z_i - z_j)(Z_i - z_j)} \subset \sum_{j \neq i} \frac{w_j}{z_i - z_j} \left\{ \frac{1}{z_i - z_j}; \frac{r}{\rho^2} \right\} \\ &\subseteq \left\{ \sum_{j \neq i} \frac{w_j}{(z_i - z_j)^2}; \frac{r}{\rho^2} \sum_{j \neq i} \frac{|w_j|}{|z_i - z_j|} \right\} = \left\{ g_{2,i}; \frac{r}{\rho^2} \sum_{j \neq i} \frac{|w_j|}{|z_i - z_j|} \right\}. \ \blacksquare \end{aligned}$$

Assume that the disjoint disks $Z_i^{(0)} = \{z_i^{(0)}; r_i^{(0)}\}$, containing the zeros ζ_i $(i = 1, \ldots, n)$, have been found. Relation (7) suggests a new interval method for finding, simultaneously, simple complex zeros of a polynomial.

Let $Z_i^{(m)}, z_i^{(m)} = \operatorname{mid} Z_i^{(m)}, r_i^{(m)} = \operatorname{rad} Z_i^{(m)}, \rho^{(m)}, r^{(m)}, w_i^{(m)}, g_{k,i}^{(m)}, s_i^{(m)}, S_i^{(m)}$ be the notation (introduced above) concerning the mth iterative step. Sometimes, for simplicity, we omit iteration index m in the current iterative step and use the symbol $\widehat{\ }$ (*"hat"*) for the quantities in the $(m+1)$st iterative step.

THEOREM 1. *Let the interval sequences* $\big(Z_i^{(m)}\big)$ $(i = 1, \ldots, n)$ *be defined by the iterative formula*

$$Z_i^{(m+1)} = z_i^{(m)} - \frac{2w_i^{(m)}}{1 + g_{1,i}^{(m)} + \sqrt{\big(1 + g_{1,i}^{(m)}\big)^2 + 4w_i^{(m)} S_i^{(m)}}} \qquad (i \in I_n;\ m = 0, 1, \ldots). \quad (12)$$

Then, under the condition

$$\rho^{(0)} > 4(n-1)r^{(0)}, \tag{13}$$

for each $i = 1, \ldots, n$ *and* $m = 0, 1, \ldots$ *we have*

1° $\zeta_i \in Z_i^{(m)}$;

2° $r^{(m+1)} < \dfrac{15(n-1)\big(r^{(m)}\big)^4}{\big(\rho^{(0)} - \frac{5}{4}r^{(0)}\big)^3}$.

Proof. We will prove the assertion 1° by induction. Suppose that $\zeta_i \in Z_i^{(m)}$ for $i \in I_n$ and $m \geq 1$. Then $s_i^{(m)} \in S_i^{(m)}$ and, on the basis of (12) and the inclusion isotonicity, it follows

$$\zeta_i \in z_i^{(m)} - \frac{2w_i^{(m)}}{1 + g_{1,i}^{(m)} + \sqrt{\left(1 + g_{1,i}^{(m)}\right)^2 + 4w_i^{(m)} S_i^{(m)}}} = Z_i^{(m+1)}.$$

Since $\zeta_i \in Z_i^{(0)}$, we obtain that $\zeta_i \in Z_i^{(m)}$ for each $m = 1, 2, \ldots$.

Let us prove now that the interval method (12) has the order of convergence equals four (assertion 2°). We use induction and start with $m = 0$. For simplicity, all indices are omitted and all quantities in the first iteration are denoted by $\widehat{\ }$.

Let us introduce the abbreviations

$$\theta = \frac{r^3}{\rho^3} \cdot \frac{4(n-1)\alpha_n^2}{(1 - \alpha_n/4)^2}, \quad \eta = 1 + \frac{4w_i g_{2,i}}{(1 + g_{1,i})^2},$$
$$\lambda = 11(n-1)\frac{r^3}{\rho^3}, \quad \varepsilon = \frac{\theta}{|\eta|^{1/2} + \left(|\eta| - \theta\right)^{1/2}}.$$

Using (13) and the estimates from Lemma 1, we can find the bounds for these quantities.

First, since $f(\alpha_n) := \alpha_n^2/(1 - \alpha_n/4)^2$ is monotonically increasing on the interval $(0, e^{1/4})$, we have $f(\alpha_n) < e^{1/2}/(1 - e^{1/4}/4)^2 \approx 3.576$ so that

$$\theta = \frac{r^3}{\rho^3} \cdot \frac{4(n-1)\alpha_n^2}{(1 - \alpha_n/4)^2} < 15(n-1)\frac{r^3}{\rho^3} < 0.06 \quad \text{for } n \geq 3. \tag{14}$$

Furthermore, by (iv) of Lemma 1,

$$|\eta| = \left|1 + \frac{4w_i g_{2,i}}{(1 + g_{1,i})^2}\right| > 1 - \frac{4|w_i g_{2,i}|}{|1 + g_{1,i}|^2} > 1 - 4 \cdot \frac{1}{8} = \frac{1}{2}. \tag{15}$$

Using (13) and the bounds (14) and (15) for θ and η, we find for $n \geq 3$

$$\varepsilon = \frac{\theta}{|\eta|^{1/2} + \left(|\eta| - \theta\right)^{1/2}} < \frac{15(n-1)(r^3/\rho^3)}{\sqrt{0.5} + \sqrt{0.5 - 0.06}} < 11(n-1)\frac{r^3}{\rho^3} = \lambda < 0.05. \tag{16}$$

Starting from the iterative formula (12) and taking into account Lemma 2 and the assertions (i) and (iii) of Lemma 1, we find

$$\begin{aligned}
\widehat{Z}_i &\subset z_i - \frac{2w_i}{(1 + g_{1,i})\left(1 + \sqrt{1 + \dfrac{4w_i}{(1 + g_{1,i})^2}\left\{g_{2,i}; \dfrac{\alpha_n(n-1)r^2}{\rho^3}\right\}}\right)} \\
&= z_i - \frac{2w_i}{(1 + g_{1,i})\left(1 + \sqrt{\left\{\eta; \dfrac{4\alpha_n(n-1)|w_i|r^2}{|1 + g_{1,i}|^2\rho^3}\right\}}\right)} \\
&\subseteq z_i - \frac{2w_i}{(1 + g_{1,i})\left(1 + \sqrt{\left\{\eta; \dfrac{r^3}{\rho^3} \cdot \dfrac{4(n-1)\alpha_n^2}{(1 - \alpha_n/4)^2}\right\}}\right)},
\end{aligned}$$

that is

$$\widehat{Z}_i \subset z_i - \frac{2w_i}{(1+g_{1,i})\Big(1+\{\eta;\theta\}^{1/2}\Big)}. \tag{17}$$

Let us note that, according to (14) and (15), it follows $0 \notin \{\eta;\theta\}$. We use (3) and find

$$\{\eta;\theta\}^{1/2} = \Big\{\eta^{1/2}; |\eta|^{1/2} - (|\eta|-\theta)^{1/2}\Big\} = \left\{\eta^{1/2}; \frac{\theta}{|\eta|^{1/2} + \big(|\eta|-\theta\big)^{1/2}}\right\} = \{\eta^{1/2};\varepsilon\}.$$

Using this result, from (16) and (17) we obtain

$$\widehat{Z}_i \subset z_i - \frac{2w_i}{(1+g_{1,i})\Big(1+\{\eta^{1/2};\varepsilon\}\Big)} \subset z_i - \frac{2w_i}{(1+g_{1,i})\Big(\Big\{1+\eta^{1/2}; 11(n-1)\dfrac{r^3}{\rho^3}\Big\}\Big)},$$

that is

$$\widehat{Z}_i \subset z_i - \frac{2w_i}{(1+g_{1,i})\{v;\lambda\}}, \tag{18}$$

where we put $v = 1+\eta^{1/2}$.

Let us prove that the inversion of the disk $\{v;\lambda\}$ appearing in (18) always exists. Let $u = 4w_i g_{2,i}/(1+g_{1,i})^2$. According to (iv) of Lemma 1 we have $|u| < 0.5$, which means that the complex number u lies in the disk $U := \{0; 0.5\}$. Since

$$v^2 = \big(1+\eta^{1/2}\big)^2 = \big(1+\sqrt{1+u}\,\big)^2 = 2+u+2\sqrt{1+u},$$

by the inclusion isotonicity and (3) we obtain

$$\begin{aligned} v^2 \in V &:= 2+U+2(1+U)^{1/2} = \{2;0.5\} + 2\{1;0.5\}^{1/2} = \{2;0.5\} + 2\{1; 1-\sqrt{1-0.5}\} \\ &\subset \{2;0.5\} + 2\{1;0.3\} = \{4;1.1\}. \end{aligned}$$

Hence

$$|v|^2 > |\mathrm{mid}\, V| - \mathrm{rad}\, V = 4 - 1.1 = 2.9. \tag{19}$$

Therefore, $|v| > \sqrt{2.9} > 1/20 > \lambda$ (according to (16)) so that $0 \notin \{v;\lambda\}$ and the complex-valued set on the right hand side of (18) is a closed disk. Using (1) we find from (18)

$$\widehat{Z}_i \subset z_i - \frac{2w_i}{1+g_{1,i}} \left\{ \frac{\bar{v}}{|v|^2-\lambda^2}; \frac{\lambda}{|v|^2-\lambda^2} \right\}.$$

From the last inclusion we find the upper bound of the radius $\hat{r}_i$ of the disk $\widehat{Z}_i$:

$$\hat{r}_i = \mathrm{rad}\, \hat{Z}_i < \frac{2|w_i|}{|1+g_{1,i}|} \cdot \frac{\lambda}{|v|^2-\lambda^2}. \tag{20}$$

We have proved that $\lambda < 1/20$ (see (16)). Using this bound, the estimates (i) and (iii) of Lemma 1 and (19), from (20) we obtain

$$\hat{r}_i < \frac{2\alpha_n r}{1-\alpha_n/4} \cdot \frac{11(n-1)r^3/\rho^3}{2.9-0.05^2}.$$

Hence

$$\hat{r}_i < 15(n-1)\frac{r^4}{\rho^3} \tag{21}$$

and, using (13),

$$\hat{r}_i < 15(n-1)r\Big(\frac{r}{\rho}\Big)^3 < \frac{15}{64(n-1)^2}r < \frac{r}{17}. \tag{22}$$

According to a geometric construction and the fact that the disks $Z_i^{(m)}$ and $Z_i^{(m+1)}$ must have at least one point in common (the zero ζ_i), the following relation can be derived (see [8]):

$$\rho^{(m+1)} \geq \rho^{(m)} - r^{(m)} - 3r^{(m+1)}. \tag{23}$$

Using the inequalities (22) and (23) (for $m = 0$), we find

$$\rho^{(1)} \geq \rho^{(0)} - r^{(0)} - 3r^{(1)} > 4(n-1)r^{(0)} - r^{(0)} - 3\,\frac{r^{(0)}}{17} > 17r^{(1)}\big(4(n-1) - 1 - 3/17\big),$$

wherefrom it follows

$$\rho^{(1)} > 4(n-1)r^{(1)}. \tag{24}$$

This is the condition (9) for the index $m = 1$, which means that all assertions of Lemmas 1 and 2 are valid for $m = 1$.

Using the definition of ρ and (24), for arbitrary pair of indices i, j $(i \neq j)$ we have

$$|z_i^{(1)} - z_j^{(1)}| \geq \rho^{(1)} > 4(n-1)r^{(1)} > 2r^{(1)} \geq r_i^{(1)} + r_j^{(1)}. \tag{25}$$

Therefore, in regard to (5), the disk $Z_1^{(1)}, \dots, Z_n^{(1)}$ produced by (12) are disjoint.

Applying induction with the argumentation used for the derivation of (21), (22), (24) and (25) (which makes the part of the proof with respect to $m = 1$), we prove that, for each $m = 0, 1, \dots$, the disks $Z_1^{(m)}, \dots, Z_n^{(m)}$ are disjoint and the following relations are true:

$$r^{(m+1)} < \frac{15(n-1)\big(r^{(m)}\big)^4}{\big(\rho^{(m)}\big)^3}, \tag{26}$$

$$r^{(m+1)} < \frac{r^{(m)}}{17}, \tag{27}$$

$$\rho^{(m)} > 4(n-1)r^{(m)}. \tag{28}$$

These inequalities follow from (21), (22) and (24), respectively. In addition we note that the last inequality (28) means that the assertions of Lemmas 1 and 2 hold in each iterative step m.

By the successive application of (23) and (27) we obtain

$$\begin{aligned}
\rho^{(m)} &> \rho^{(m-1)} - r^{(m-1)} - 3\,\frac{r^{(m-1)}}{17} = \rho^{(m-1)} - r^{(m-1)}(1+3/17) \\
&> \rho^{(m-2)} - r^{(m-2)} - 3\,\frac{r^{(m-2)}}{17} - \frac{r^{(m-2)}}{17}(1+3/17) \\
&= \rho^{(m-2)} - r^{(m-2)}\Big(1 + \frac{4}{17} + \frac{4}{17^2} - \frac{1}{17^2}\Big) \\
&\ \vdots \\
&> \rho^{(0)} - r^{(0)}\Big(1 + \frac{4}{17} + \frac{4}{17^2} + \cdots + \frac{4}{17^{m-1}} - \frac{1}{17^{m-1}}\Big) \\
&> \rho^{(0)} - \frac{5}{4} r^{(0)}.
\end{aligned}$$

According to the last inequality and (26) we find

$$r^{(m+1)} < \frac{15(n-1)\big(r^{(m)}\big)^4}{\big(\rho^{(0)} - \frac{5}{4} r^{(0)}\big)^3}. \tag{29}$$

Strict inequality in (29) indicates that the order of convergence of the inclusion method (12) is at least four. However, from the assertions (i) and (ii) of Lemma 2 we see that $|w_i| = O(r)$ and $|g_{k,i}| = O(r)$. Besides, from (10) it follows $\operatorname{rad} S_i = O(r^2)$. These estimations give in the final result only $r^{(m+1)} = O\big((r^{(m)})^4\big)$, which means that the order of convergence of (12) is precisely four. ■

The convergence of the Euler-like inclusion method (8) can be increased without additional calculations using an approach presented in the papers [9], [10] and [11]. First we show that the inequality (13) guarantees the implication $\zeta_i \in Z_i \Rightarrow \zeta_i \in Z_i - w_i$ in each iteration. Taking the removed disk $Z_i - w_i$ instead of Z_i in (8), the following algorithm has been stated in [4]:

Algorithm 2: *Let* $(Z_1, \ldots, Z_n) =: \big(Z_1^{(0)}, \ldots, Z_n^{(0)}\big)$ *be initial disks such that* $\zeta_i \in Z_i$ $(i = 1, \ldots, n)$. *One step of the modified Euler-like inclusion algorithm reads* $(Z_1, \ldots, Z_n) \mapsto (\hat{Z}_1, \ldots, \hat{Z}_n)$ *with*

$$\hat{Z}_i := z_i - 2w_i\,\mathrm{INV}_1\left(1 + g_{1,i} + \sqrt{(1+g_{1,i})^2 + 4w_i \sum_{j\neq i} \frac{w_j\,\mathrm{INV}_2(Z_i - w_i - z_j)}{z_i - z_j}}\right), \tag{30}$$

where INV_1 *and* INV_2 *denote inversions of a disk given by* (1) *and* (2), *that is,* $\mathrm{INV}_1, \mathrm{INV}_2 \in \{()^{-1}, ()^I\}$.

Using the concept of the R-order of convergence introduced by Ortega and Rheinboldt [12] it can be proved that the R-order of convergence of the radii for the inclusion method (30) is at least $2 + \sqrt{7} \cong 4.646$ if $\mathrm{INV}_2 = ()^{-1}$ and 5 otherwise.

The proof of this assertion will be given in the forthcoming paper. As noted in [4], the increase of the convergence rate of Algorithm 2 in reference to Algorithm 1 is forced by the very fast convergence of the sequences $\{z_i^{(m)}\}$ of the centers of disks, which converge with the convergence rate *fifth.* This acceleration of convergence is attained since the better approximation $\operatorname{mid}(Z_i - w_i) = z_i - w_i$ to the zero ζ_i is used instead of the former $\operatorname{mid} Z_i = z_i$.

4. THE CHOICE OF INITIAL DISKS

In practice, a complete procedure for improving the approximations to the zeros of a polynomial consists of a four-stage *globally convergent* algorithm:

(a) Find an inclusion region of the complex plane which includes all zeros of a given polynomial $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$. A region with this property is the disk centered at the origin with the radius

$$\sigma = 2 \max_{1 \le k \le n} |a_{n-k}|^{1/k}.$$

It is possible to use other similar formulas, but the disk $\{0;\sigma\}$ has been found to be sufficient and satisfactory in practice.

(b) Apply a slowly convergent search algorithms to obtain a crude disjoint initial intervals (disks or rectangles) which contain polynomial zeros.

(c) Improve crude inclusion intervals to obtain separated intervals, each containing one zeros. These intervals should provide a convergence of a fast interval method in the next stage.

(d) Improve intervals found in the stage (b) by applying a rapidly convergent iterative interval method to any required accuracy.

Useful results concerning the stages (a) and (b) can be found in [1], [13], [14], [15] and references cited therein. A number of inclusion methods with fast convergence, mentioned in the stage (d), was considered in the book [1]. The inclusion method analyzed in this paper is also one of fast convergent methods. In this section we give an efficient procedure for finding initial disks, which can be regarded as the stage (c) in the frame of the described global algorithm.

The choice of initial regions is often closely connected with initial conditions for the convergence of iterative methods. Most of these conditions treated in literature are not of practical importance since they depend on unknown data; for example, initial conditions deal with (vague) constants or involve some functions of the wanted zeros. In this section we give in short a practically applicable procedure for the choice of initial inclusion disks which provide the safe convergence of inclusion methods including, in particular, the Euler-like interval method (12). These disks depend on available initial data and their construction is based on the recent results given in [16].

Let $z_1^{(0)}, \dots, z_n^{(0)}$ be distinct points in the complex plane and let

$$d^{(0)} = \min_{\substack{i,j \\ j \ne i}} |z_i^{(0)} - z_j^{(0)}|, \quad w(z_i^{(0)}) = \frac{P(z_i^{(0)})}{\prod\limits_{\substack{j=1 \\ j\ne i}}^{n} \bigl(z_i^{(0)} - z_j^{(0)}\bigr)}, \quad w^{(0)} = \max_{1 \le i \le n} w(z_i^{(0)}).$$

The following assertion has been proved in [16, §1.2] (indices are omitted):

THEOREM 2. *Let $c_n := 1/(\alpha n+\beta)$, $\alpha \geq 2$, $\beta > (2-\alpha)n$, and let us assume that $w < c_n d$ holds. Then for $n \geq 3$ the disks*

$$D_i := \Big\{z_i; \frac{1}{1-nc_n}|w_i|\Big\} \quad (i \in I_n)$$

are mutually disjoint and each of them contains one and only one zero of P.

Taking $r_i = \operatorname{rad} D_i = |w_i|/(1-nc_n)$ and using the above notation, under the condition $w < c_n d$ we obtain

$$\begin{aligned}\rho_{ij} &:= |z_i - z_j| - r_i = |z_i - z_j| - \frac{1}{1-nc_n}|w_i| > d - \frac{c_n d}{1-nc_n} = d\Big(1 - \frac{c_n}{1-nc_n}\Big)\\ &> \frac{w}{c_n}\Big(1-\frac{c_n}{1-nc_n}\Big) = \frac{1-(n+1)c_n}{c_n}r.\end{aligned}$$

Since this inequality holds for arbitrary pair of indices (i,j) $(i \neq j)$ and therefore, it will hold for the pair (i_0, j_0) such that $\rho_{i_0 j_0} = \min_{i\neq j} \rho_{ij} = \rho$, we have proved the implication

$$w < c_n d \quad \Rightarrow \quad \rho > \frac{1-(n+1)c_n}{c_n} r. \tag{31}$$

In the concrete case of the presented Euler-like method (12) we have to choose the constant c_n so that the first inequality in (31) implies the inequality

$$\frac{1-(n+1)c_n}{c_n} > 4(n-1).$$

(according to (13)). It is easy to check that the choice $c_n = 1/5n$ provides the validity of the last inequality. For this specific value of c_n we find that the radius r_i of the disk D_i in Theorem 2 is $r_i = \frac{5}{4}|w_i|$. In this way we have proved the following theorem:

THEOREM 3. *Let $z_1^{(0)}, \ldots, z_n^{(0)}$ be initial distinct approximations to the simple zeros $\zeta_1, \ldots, \zeta_n$, and let*

$$w^{(0)} = \max_{1 \leq 1 \leq n} |w(z_i^{(0)})| < \frac{d^{(0)}}{5n}.$$

Then the Euler-like method (12) *converges starting with the initial disjoint disks*

$$\{z_1^{(0)}; \tfrac{5}{4}|w(z_1^{(0)})|\}, \ldots, \{z_n^{(0)}; \tfrac{5}{4}|w(z_n^{(0)})|\}$$

which contain the zeros $\zeta_1, \ldots, \zeta_n$, respectively.

We note that the disks given in Theorem 3 are considerably smaller than Smith's disks $\{z_i - w_i; (n-1)|w_i|\}$ given in [17] and the disks $\{z_i; n|w_i|\}$ constructed in [18].

The initial condition (13) is only sufficient; the Euler-like method (12) can converge even if the initial condition (13) is not satisfied, which has been confirmed in a number of numerical examples. Considering Theorem 2 we can conclude that the choice of a greater c_n gives greater inclusion disks $D_i = |w_i|/(1-nc_n)$ and a smaller quantity $(1-(n+1)c_n)/c_n$ in the inequality (31). In this way the initial conditions are weakened, but practical experiments shows that most of inclusion methods, including Euler-like method (12), still remain convergent until initial disks are disjoint. More details about practical realization of the Euler-like inclusion method (12) and numerical experiments can be found in [4].

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