Cayley's Problem and Julia Sets

H. O. Peitgen, D. Saupe, and F. v. Haeseler

Blindness to the aesthetic element in mathematics is widespread and can account for a feeling that mathematics is dry as dust, as exciting as a telephone book, as remote as the laws of infangthief of fifteenth century Scotland. Contrariwise, appreciation of this element makes the subject live in a wonderful manner and burn as no other creation of the human mind seems to do.

P. J. Davis and R. Hersch

My work has always tried to unite the true with the beautiful and when I had to choose one or the other I usually chose the beautiful.

H. Weyl

Experimental mathematics will likely never be accepted as "real" mathematics by most mathematicians. But for many enthusiasts it has become more than an engaging hobby—it is rather a passion. While such experiments will continue to enhance our mathematical intuition in the future, they might also develop into a sophisticated art form.

When, in the spring of 1983, we made our first discoveries about Julia Sets in a computer graphics lab, we were quite ignorant about the mathematical beauty and the depth of the subject. Since that time we have become addicted, and our addiction has led us into a beautiful area of mathematics.

The goal of this exposition is to give a flavor of the subject of Julia Sets which we trace back to a problem posed by Arthur Cayley in 1879. Our computer graphics not only illustrated the beauty that can be found in Julia sets, but they also provided us with insight that led us to some new results. In this exposition we can give only a short introduction; in fact, this is an excerpt from a longer article entitled "Newton's Method and Julia Sets" which contains more of the background mathematics.

Cayley's Problem

In 1879 A. Cayley [3] suggested the extension of what he called the Newton-Fourier Method

$$N(\zeta_k) := \zeta_{k+1} = \zeta_k - p(\zeta_k)/p'(\zeta_k)$$

to complex roots of a polynomial p:

. . . In connexion herewith, throwing aside the re-

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strictions as to reality, we have what I call the Newton-Fourier Imaginary Problem . . . .

Furthermore, he suggested that one should study the problem globally:

. . . The problem is to determine the regions of the plane, such that P [initial point] being taken at pleasure anywhere within one region we arrive ultimately at the point A [a root of the polynomial] . . . .

In two notes published in 1879 [4] and 1890 [5] he took up the problem for $p(z) = z^2 - 1 = 0$ in C. More generally, if $p(z) = a(z - z_1) \ldots (z - z_n)$, then each root z_i of p is an attractive fixed point of the dynamical system N. Locally, near z_i convergence is quadratic if z_i is a simple root (i.e., $p'(z_i) \neq 0$). Let

$$A(z_i) = \{z \in C : N^n(z) \to z_i,\ n \to \infty\}$$

be the i-th basin of attraction ($N^n = N\circ \ldots \circ N$, n–times). Then Cayley asked: What are the sets $A(z_k)$ and what is the boundary $\partial A(z_k)$?

It turns out to be worthwhile to study these questions for the simple model $p(z) = z^2 - 1$. The reader may have guessed already that in this case

$$A(+1) = \{z : Re(z) > 0\},$$
$$A(-1) = \{z : Re(z) < 0\}$$

and that

$$\partial A(+1) = \partial A(-1) =: J,$$

i.e., J is the imaginary axis. Note that

$$N(J) = J = N^{-1}(J),$$

where $N^{-1}(J)$ denotes the set of preimages of J. Thus, the restriction of N to J reduces to the 1–dimensional dynamical system

$$\alpha i \to \frac{1}{2}\left(\alpha - \frac{1}{\alpha}\right) i,\ \alpha \in \mathbf{R}.$$

The reader may wonder if this peculiar behaviour is due to the simplicity of this particular p. The answer is no; if p is any polynomial of degree 2 then first by a change of coordinates we can reduce it to $p_\lambda(z) = z^2 - \lambda$ and then it is easy to see that for the line $J_\lambda := \{\alpha i\sqrt{\lambda} : \alpha \in \mathbf{R}\}$ we have that $N_\lambda(J_\lambda) = J_\lambda = N_\lambda^{-1}(J_\lambda)$, where N_λ is Newton's Method applied to p_λ. Thus, we guess that J_λ bounds the two basins of attraction, which, as we will see is the case.

Let's give a simple argument for the case $\lambda = 1$. It's easier to see what is happening if we make a change of variable. Let T be the linear (Möbius) transformation

$$T(z) = \frac{z-1}{z+1} \text{ with } T^{-1}(u) = \frac{1+u}{1-u},$$

which we consider in $\Sigma = C \cup \{\infty\}$. Then we obtain that

$$R(u) = u^2 \text{ where } R(u) := T \circ N \circ T^{-1}(u).$$

Here is a list of interesting points

z	$+1$	-1	∞	0
$T(z)$	0	∞	1	-1

The imaginary axis J corresponds to the unit circle S^1 under the transformation T. Thus, we have answered Cayley's question, because 0 and ∞ are the attractors of R i.e., $|R'(0)| < 1$, and their basins of attraction are separated by S^1. For curiosity we may add in passing that we can interpret $R(z) = z^2$ as Newton's Method for the rational function $f(z) = z/(1-z)$.

In Cayley's paper [5] the transformation T is also mentioned, but he prefers a somewhat different representation of N with a certain hope in his mind:

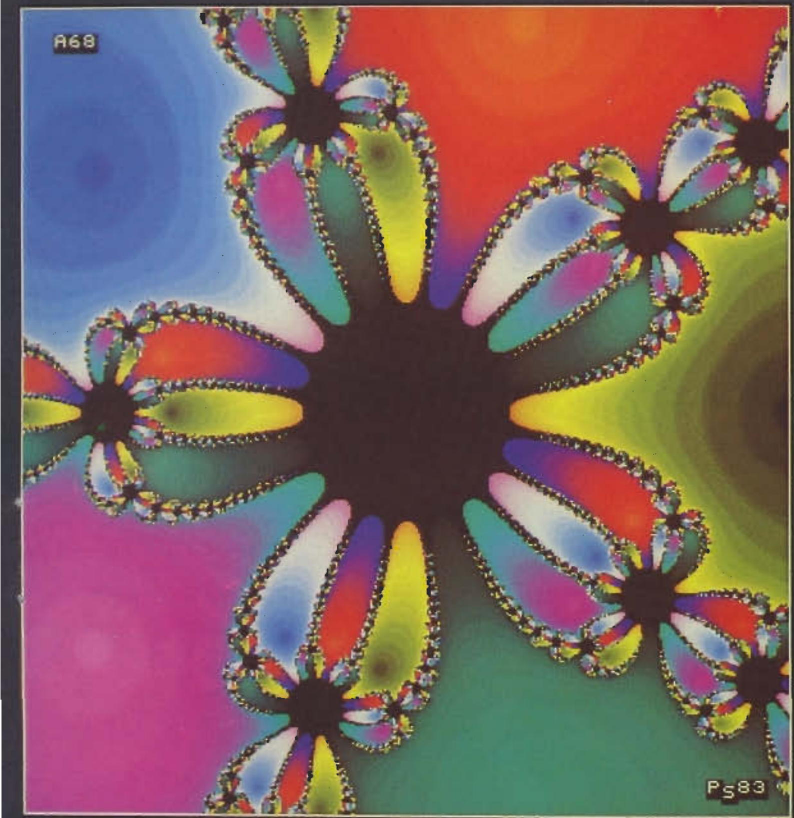
J'espère appliquer cette théorie au cas d'une équation cubique, mais les calculs sont beaucoup plus difficiles.

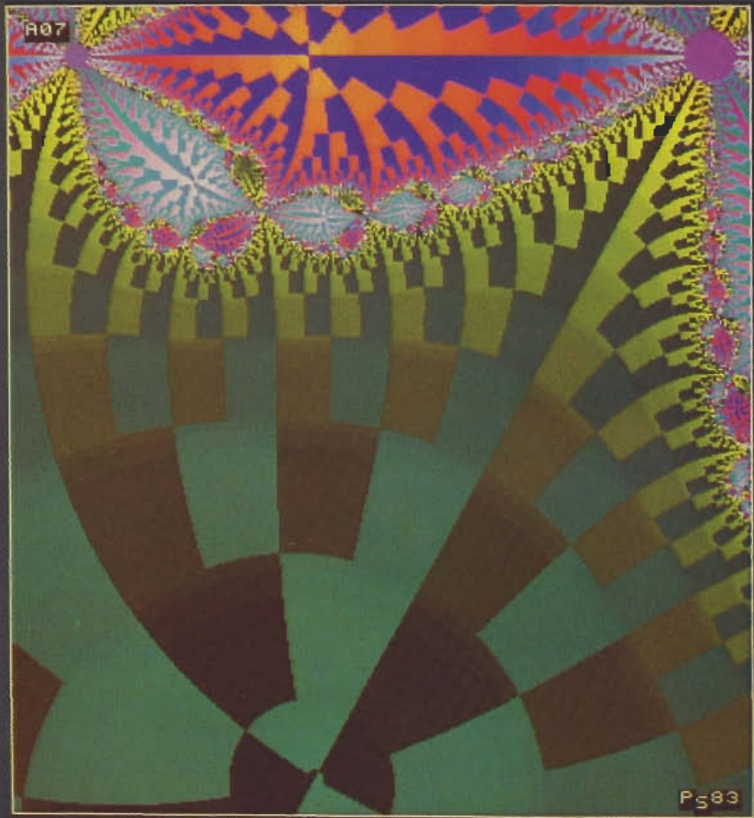
Maybe he already knew that for the cubic case one cannot expect to have such an elegant global conjugation as in the quadratic case. His hope was not fulfilled, and it took almost 30 years until Julia and Fatou developed their fascinating theory of the iteration of rational functions in the complex plane which explains why Cayley's problem is such a hard one. Julia's work [10] was rewarded with the "Grand Prix des Sciences mathématiques" by the Académie des Sciences de Paris.

Before we state some of these classical results, we want to discuss a few of them for the model case $R(z) = z^2$ and $J = \{z : |z| = 1\}$. Recall that $\partial A(0) = J = \partial A(\infty)$ and note that if $R^n(z) = z^{2^n}$, then in $A(0)$ (resp. $A(\infty)$) the limit of R^n exists and is the constant 0 (resp. ∞). However, if $z \in J$, then in any neighborhood of z neither R^n nor any subsequence has an analytic limit. Sets with such abnormal properties were the focus of attention of Julia [10] and are now called Julia sets. A formal definition is this: Let $R: \Sigma \to \Sigma$ be a rational function of degree greater or equal than two on the Riemannian sphere Σ. We say that R is *normal* for a point $z \in \Sigma$, provided there exists a neighborhood U of z such that the sequence $\{R^n|_U\}_{n \in N}$ of mappings from U to Σ is equicontinuous.

The set J of points in Σ for which R is not normal is called the *Julia set* of R.

For a discussion of normal families we refer to [1]. In the model case $R(z) = z^2$ it is clear that the Julia set J is the unit circle S^1. Also we have again that J is completely invariant, i.e., $R(J) = J = R^{-1}(J)$. R as a map-





ping from Σ into itself has 3 fixed points, 0 and ∞, which are attractors, and $1 \in J$, which is a repeller, i.e., $|R'(1)| > 1$. For the purpose of understanding Julia sets it is the repeller, which is most crucial to understand. Two facts are immediate:

1. The set of the preimages of the repeller $S = \{z : R^n(z) = 1 \text{ for some } n \in \mathbf{N}\} = \{\exp(2\pi i\alpha) : \alpha = k/2^n, k, n \in \mathbf{N}\}$ is dense in J.
2. The set of periodic points in J $P = \{z \in J : R^n(z) = z \text{ for some } n \in \mathbf{N}\} = \{\exp(2\pi i\alpha) : \alpha = k/(2^n - 1), k, n \in \mathbf{N}\}$ is dense in J.

Moreover, note that each of the periodic points $R^n(z) = z \in P$ is a periodic repeller (i.e., $|(R^n)'(z)| = 2^n > 1$), and again the set of preimages of such a periodic repeller is dense in J. We remark that the dynamics of R on J is much more delicate (see [14]), e.g., R is ergodic, i.e., "most" orbits are dense.

Some Basic Facts about Julia Sets

It is surprising that most of the properties which we discovered in our digression on the map $R(z) = z^2$ are characteristic properties of Julia sets as such. Let R be any rational function on Σ, then (see [2],[6],[7],[8],[10]) if J denotes the Julia set we have

- $J \neq \emptyset$ and J is closed.
- Let z_r be a periodic repeller, then $J = cl\,\{z \in \Sigma : R^n(z) = z_r, \text{ for some } n \in \mathbf{N}\}$.
- The periodic repellers are dense in J.
- If z_a is a periodic attractor, then $z_a \notin J$.
- If $\bar{z} \in J$, then $J = cl\,\{z \in \Sigma : R^n(z) = \bar{z}, \text{ for some } n \in \mathbf{N}\}$
- $R(J) = J = R^{-1}(J)$. (1)
- The Julia sets with respect to R and with respect to R^k, $k \in \mathbf{N}$, are the same.
- If z_a is an attractive fixed point of R, then $\partial A(z_a) = J$.
- If $z \in J$ and U is any neighborhood of z, then $\{R^n(U)\}$ covers Σ except for at most two points (Montel).
- If J has interior points, then $J = \Sigma$.
- *If* D is any domain such that $D \cap J = J^* \neq \emptyset$, then there exists an integer n such $J = R^n(J^*)$.

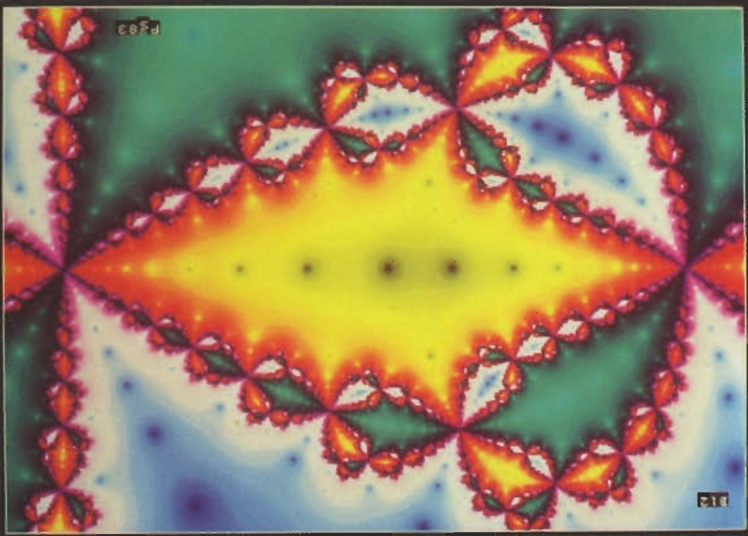
Concerning interior points it should be mentioned that in 1918 Lattès [11] gave a striking example for $J = \Sigma$: The rational function is the map $R(z) = (z^2+1)^2/4z(z^2-1)$, which turns out to be not quite as innocent as it looks at first glance. E.g., it satisfies $P(2z) = R(P(z))$, where P is a Weierstrass P-function.

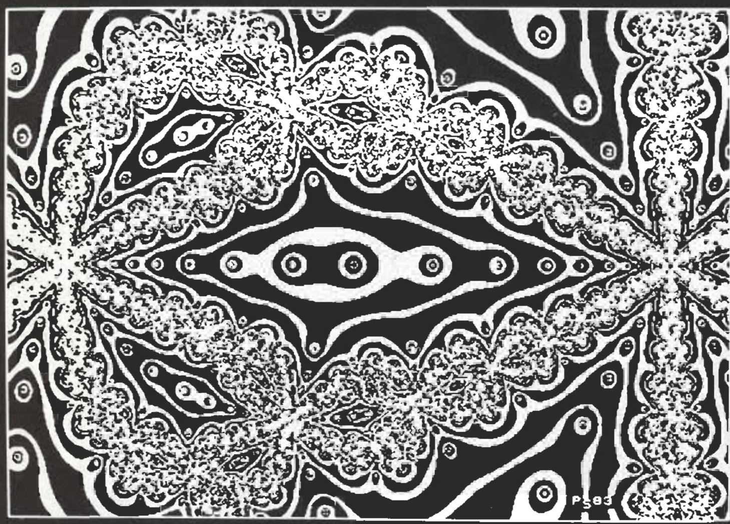
A Coloring Problem—What Was Cayley's Conjecture?

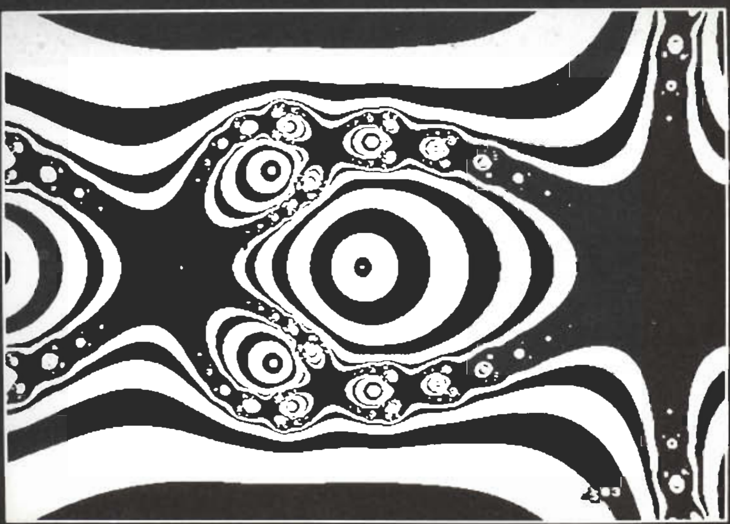
Let us now take the step from $p(z) = z^2-1$ to $p(z) = z^3-1$, which was the step Cayley had in mind. Now Newton's Method yields the rational function

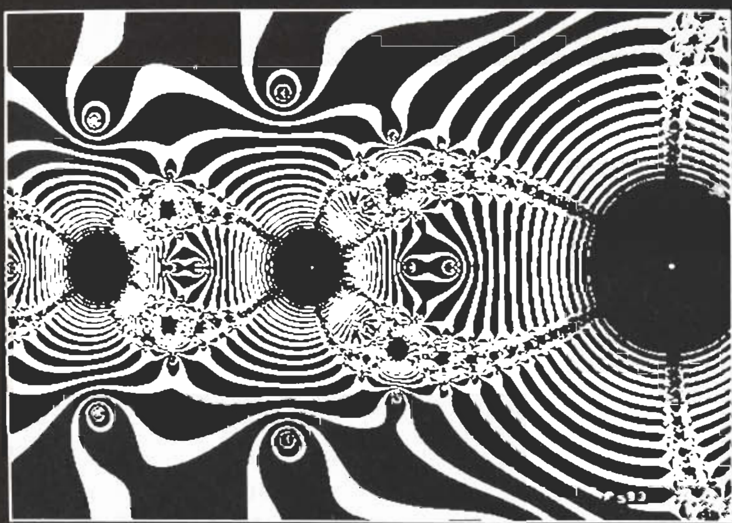
$$N(z) = z - \frac{z^3 - 1}{3z^2} = \frac{2z^3 + 1}{3z^2}$$

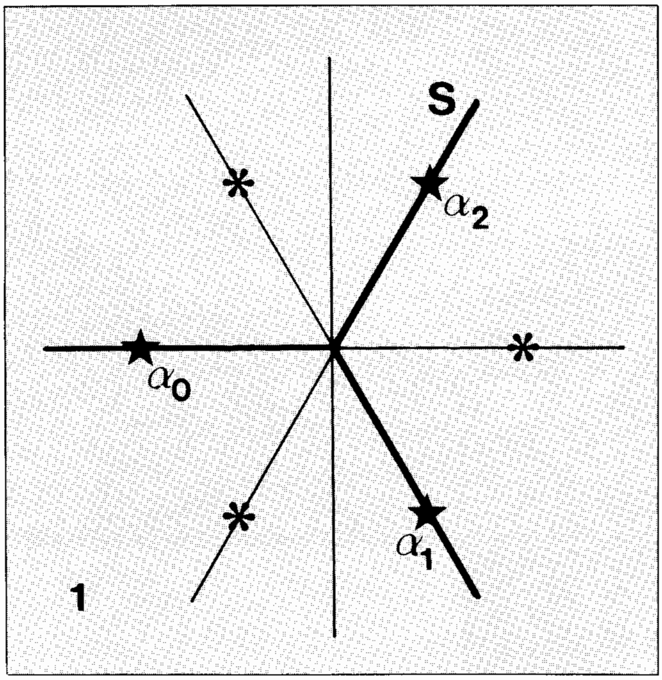
We recall that for $p(z) = z^2-1$ we had a basic symmetry with respect to the imaginary axis, which defined the basins of attraction. Naturally, one would hope that a similar idea would work here, too. Let us

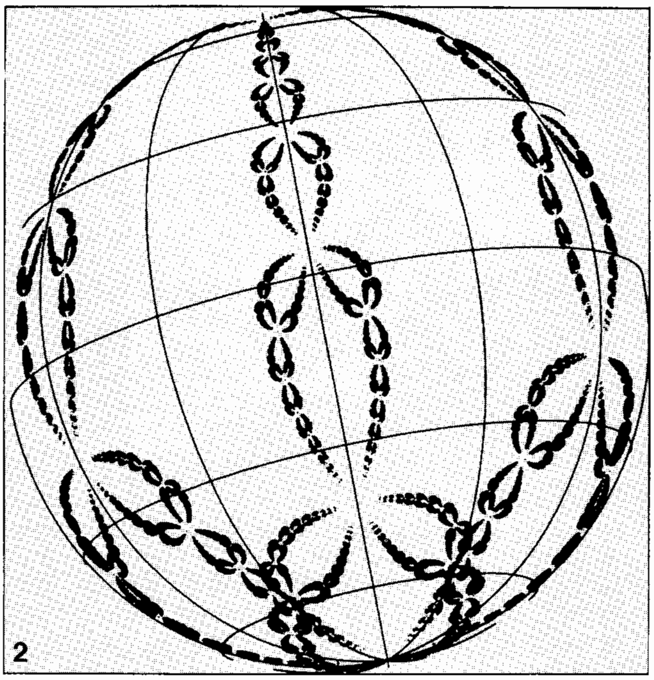












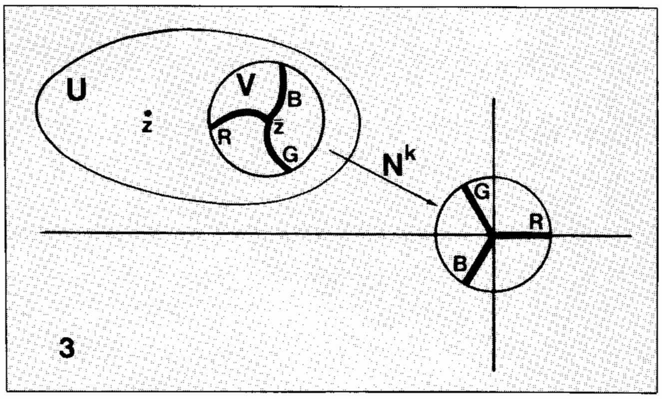


Figure 1. Is the Julia set of $N(z) = (2z^3 + 1)/3z^2$ given by the three rays through the snap back points α_k?

Figure 2. The Julia set of Newton's Method for $z^3 - 1 = 0$. Here the plane **C** of complex numbers is identified with the sphere: The South pole corresponds to 0, the North pole to ∞, and the equator corresponds to all points z with $|z| = 1$. The center branch of crabs really is on the back side of the sphere, and it has been included in the picture for completeness.

Figure 3. How the coloring problem can be solved by Newton's Method.

see, however. As with the quadratic we expect a symmetry from the roots of unity. Indeed, one computes that

$$N(z) = D \circ N \circ D^{-1}(z) \tag{2}$$

where $D(z) = \exp(2\pi i/3)z$. Moreover, we have learned that we should look for the fixed points of N

$$\mathrm{Fix}(N) = \{z_0 = 1, z_1 = (-1+\sqrt{3}i)/2, z_2 = (-1-\sqrt{3}i)/2, \infty\}$$

the first three points being attractors and the last one being a repeller. Also note that we have the snap back points 0 and α_k:

$$N(0) = \infty \text{ and}$$
$$N(\alpha_k) = 0 \text{ where}$$
$$\alpha_k = -\exp(2\pi ik/3)/\sqrt[3]{2}$$

Thus, all these points ∞, 0, $\alpha_0, \alpha_1, \alpha_2$ are in the Julia set J. Furthermore, one observes that the three lines determined by 0 and any of the α_k are invariant under N, a situation completely analogous to the case of $p(z) = z^2 - 1$. There $N(0) = \infty$, 0 has the two preimages $\pm i$ and J is the imaginary axis. Therefore, it appears to be reasonable to guess that here J is given by the set S shown in figure 1.

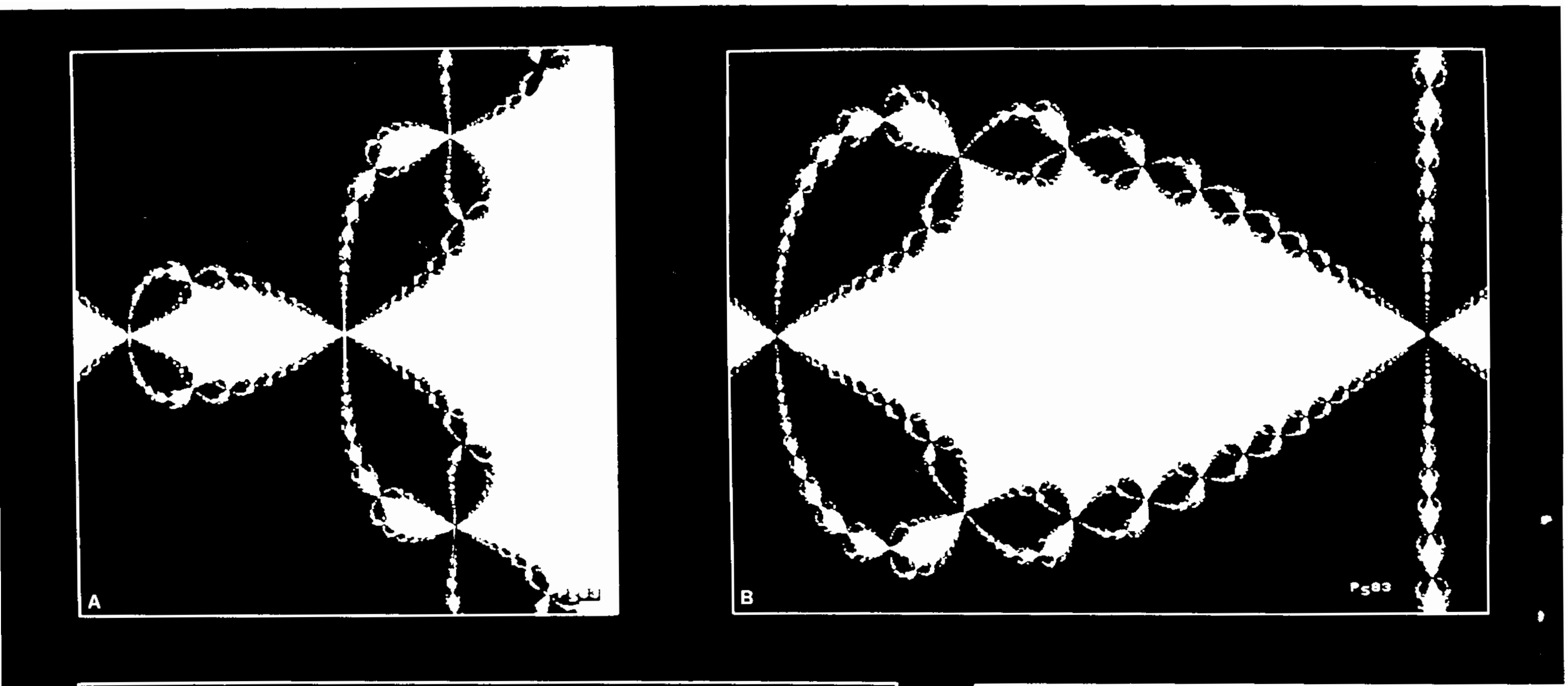
The surprise is that this guess is fundamentally wrong, and it is likely that Cayley had suspected some of the complexity shown in the front cover and figure 2, which is a very coarse approximation to J. If we would have recalled one of the earlier basic properties of J, which is that

$$J = \partial A(z_0) = \partial A(z_1) = \partial A(z_2), \tag{3}$$

then we would have never been misled. Property (3) can be illustrated by the following amazing problem: Using 3 colors try to completely color a given square, such that wherever two of the colors meet, all of them meet. After a few attempts with pencils and paper it seems that this coloring problem is not solvable. But (3) is precisely a solution to that problem.

Let us give a short argument for (3), starting from the fact that

$$J = cl\,\{z : N^n(z) = 0 \text{ for some } n \in \mathbf{N}\}. \tag{4}$$



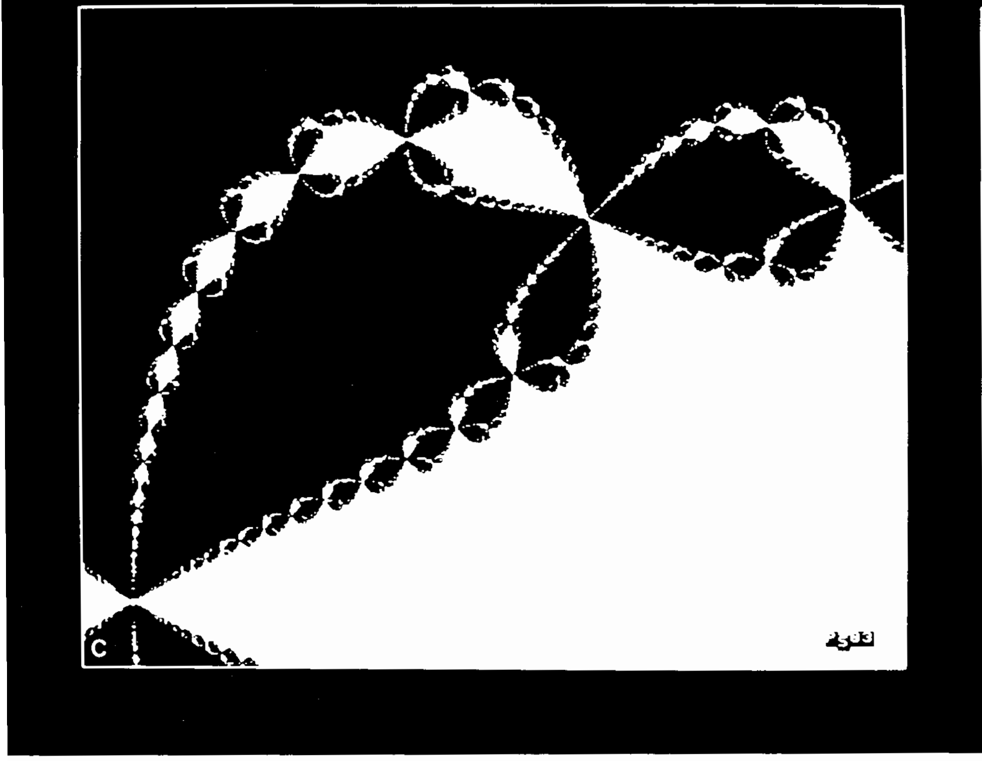


Figure 4. The complete basin of attraction belonging to the root $+1$ of $z^3 - 1 = 0$ is shown in figure 4a in white (the region again is $[-1,1] \times [-1,1]$). In b) and c) successive closeups are given revealing that the basin splits into infinitely many components. Moreover, we note the self-similarity in the pictures: The crab-like structures seem to repeat on each scale. Indeed, this is a characteristic property of Julia sets (see the last statement in (1)).

Let $z \in J$ and let U be an arbitrarily small neighborhood of z. Due to (4) we find $\bar{z} \in U$, such that $N^k(\bar{z}) = 0$ for some $k \in \mathbf{N}$. Note that $(N^k)'(\bar{z}) \neq 0$ (indeed, the only points z with $N'(z) = 0$ are the fixed points of N, i.e., the roots of p), and therefore N^k is invertible in a small neighborhood of $\bar{z}$ (see figure 3).

Hence, we find a neighborhood V of $\bar{z}$ and an ϵ-ball $B_\epsilon(0)$ around 0, such that N^k is 1-1 and onto from V to $B_\epsilon(0)$. Let I_ϵ be the interval $I_\epsilon = (0,\epsilon) \subset \mathbf{R}$. Then it is easy to see (look at the graph of $p(x) = x^3 - 1$, $x \in \mathbf{R}$) that $I_\epsilon \subset A(+1)$, and by symmetry (D as in (2)) we have $D(I_\epsilon) \subset A(z_2)$ and $D^2(I_\epsilon) \subset A(z_3)$, which proves that the three basins of attraction have points in U.

The question remains, what is wrong with the analogy to the case $p(z) = z^2 - 1$? In fact $N(S) = S$. However, as the reader has guessed already, $N^{-1}(S) \neq S$, i.e., S is *not* completely invariant.

Some Computer Graphical Experiments

Given a nonlinear rational function $R: \Sigma \rightarrow \Sigma$ with an attractor (attractive fixed point) z_a we know that $J = \partial A(z_a)$. Given also the fact that the dynamics of R on J is extremely complicated the question arises how these features are inherited to $A(z_a)$. In this context we introduce a decomposition into *level sets of equal attraction* $L_k(z_a)$:

Let $0 < \epsilon << 1$ and set $L_0(z_a) = \{z : |z - z_a| < \epsilon\}$ or, if $z_a = \infty$, then set $L_0(z_a) = \{z : |z| > \epsilon^{-1}\}$.
Then define for $k = 0,1, \ldots$
$L_{k+1}(z_a) = \{z \in \Sigma \backslash L_0(z_a) : R(z) \in L_k(z_a)\}$

As an example we have that for $R(z) = z^2$ the level sets are concentric annuli, and in an obvious sense we have that $\partial L_k \to J$ as $k \to \infty$. Most of the pictures displayed in this article are based on photographs taken from so-called raster graphic devices. One image typically consists of a matrix of about 500 by 500 dots (so called picture elements or *pixels*) each of which has a well defined color. For our purposes we identify a pixel with a point z in a rectangular region of interest in the complex plane. For each pixel we perform the iteration of an a priori selected rational function R until we have identified the basin of attraction to which the point z, resp. the pixel belongs. At the same time we determine to which level set L_k that particular pixel belongs. The remaining task is to display the pixel in an appropriate color identifying its level set.

Figures 4 and 5 deal with Newton's Method for z^3-1, in the following figure 6 we take the step to z^4-1, and in plate 1a we go on to z^5-1 and color. In the last named picture we have 5 basins of attraction, one for each of the fifth roots of unity. In each basin we show a certain number of level sets L_o to L_n in colors of our choice, whereas L_k, $k > n$ remains black.

From the material in [14] we have chosen two more experiments to be included here. The first one describes the parameter dependence of the Julia and the level sets. As a parameter we used the relaxation parameter λ in Newton's method

$$N_\lambda(z) = z - \lambda p(z)/p'(z).$$

This parameter is often employed to enlarge the domains in the basins of attraction which contain the unknown roots and to improve the rate of convergence at multiple roots, two objectives which may be offsetting each other (see [18]). In figure 7 and in plate 2 we show pictures for the overrelaxed method ($\lambda = 3/2$, top figures), the original method ($\lambda = 1$, center figures) and the underrelaxed method ($\lambda = 1/2$, bottom). The polynomial is again z^3-1.

Two observations are noteworthy: As λ increases one observes that the Julia sets J_λ become more and more complex. We would conjecture that the Hausdorff dimension of J_λ increases as λ is varied from 0.5 to 1.5. Moreover, $\lambda = 1$ is a *singular* parameter in the sense, that only for that choice the rate of convergence is quadratic near the roots. For example for the root $+1$ this means that $N'_\lambda(1) = 1 - \lambda = 0$ for $\lambda = 1$. Another way to look at this is to consider $N_\lambda(z) - 1 = 0$. For $\lambda = 1$ we have the solutions $\{1,1, -1/2\}$, i.e., 1 is a double root. For $\lambda \neq 1$ the double root 1 splits (bifurcates) into two different roots and this is visualized in the figures where one observes that the basins of attraction show additional structure for $\lambda = 0.5$ and $\lambda = 1.5$. While, e.g., in the central domain shown in plate 2 (center) there is only one point $\bar{z}$ such that $N^k{}_1(\bar{z}) = 1$, $k \in \mathbf{N}$, which is $\bar{z} = -1/2$ for $k = 1$, the other figures indicate that for $\lambda = 1.5$ and $\lambda = 0.5$ there are infinitely many such points (find the "mountain tops").

Figure 5. The clown. Allowing our phantasy to construct associations for the computer images, we may find fairy tale figures, animals or landscapes in the pictures. The above "clown" is in this spirit. Note, that in his face we can find even more clowns. It is the result of coloring the level sets in figure 4b alternating black and white and turning the picture sidewards up.

Finally, it is noteworthy that the image data in each row of plate 2 is absolutely the same. The only difference is in the color of level sets.

The last experiment in this paper introduces a further decomposition of the level sets.

Binary decompositions and partial conjugation

As before, let z_a be an attractor for the rational map R, and let L_k, $k = 0,1, \ldots$ be corresponding level sets. Their *binary decomposition* is defined in the following way:

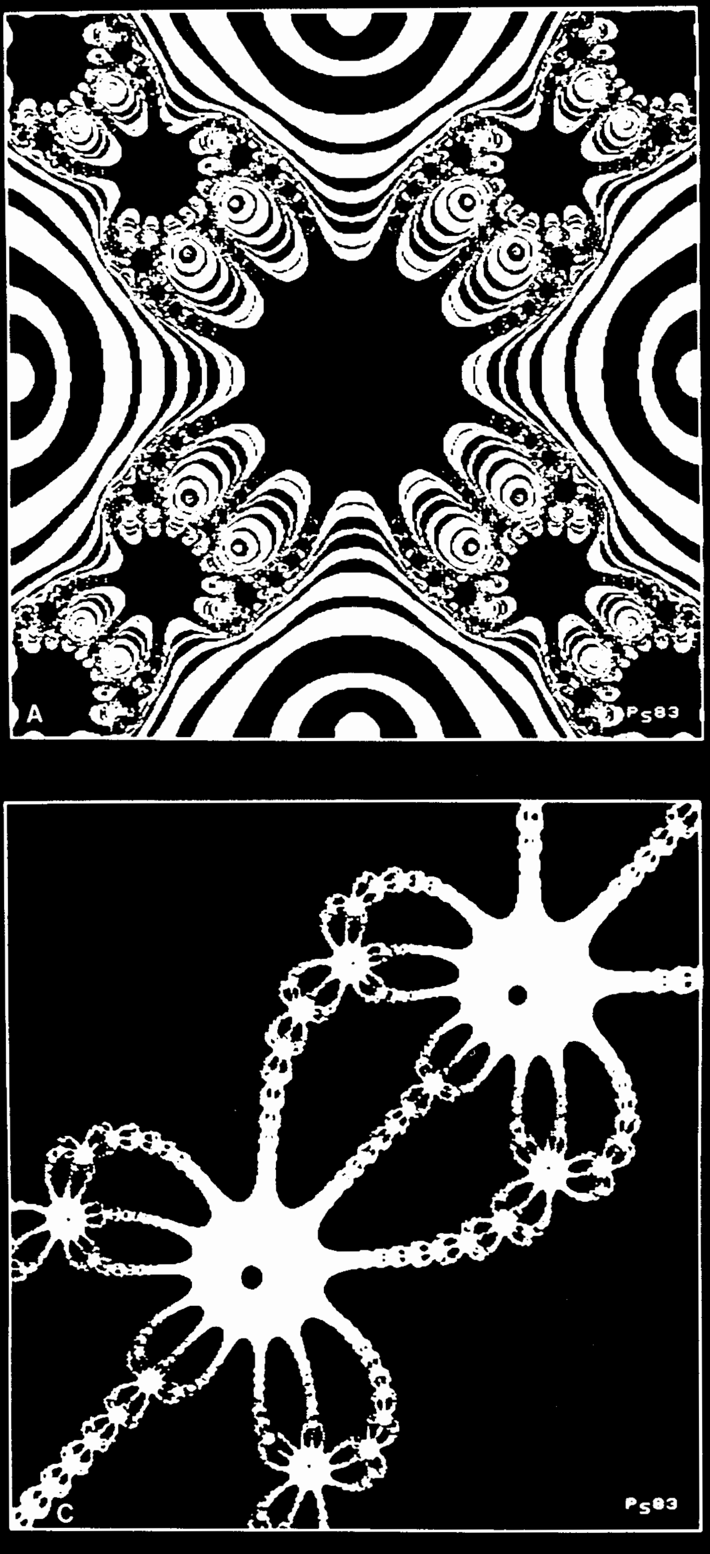


Figure 6. Newton's Method for $z^4 - 1 = 0$. In b) and c) only the upper right branch is shown.

Choose disjoint and nonempty sets L_{oo} and L_{o1} such that $L_o = L_{oo} \cup L_{o1}$. Then define for $l = 0,1$ and $k = 1,2, \ldots$

$$L_{kl} = \{z \in L_k : R^k(z) \in L_{o1}\}.$$

Thus, we have subdivided each level set into two subsets. As an example we choose for $R(z) = z^2$ $L_{oo}(z_a) = \{z \in L_o(z_a) : \text{Imag}(z) < 0\}$ and $L_{o1}(z_a) = L_o(z_a) \backslash L_{oo}(z_a)$ where $z_a = 0, \infty$ (see figure 8).

For other maps R we can choose other binary decompositions of the generating level sets L_o. E.g., in the Newton Method for $p(z) = z^3 - 1$ we split $L_o(z_o)$ as for $R(z) = z^2$ into the parts above and below the real line and use the underlying symmetry D (2) to define the binary decomposition of the other level sets $L_{o1}(z_m)$, $l = 0,1$, $m = 1,2$:

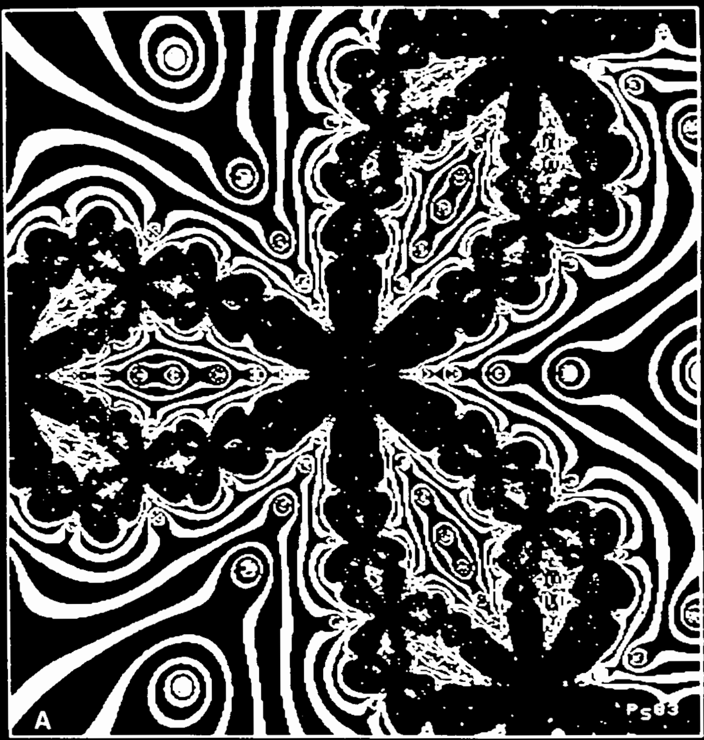
$$L_{o1}(z_m) = D^m(L_{o1}(z_o)).$$

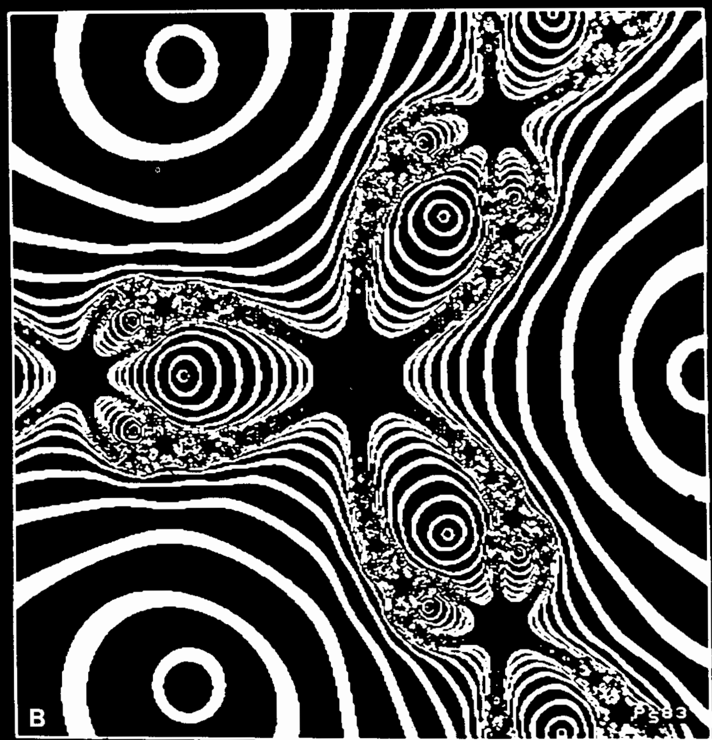
As results with colors "taken at pleasure" we obtain the front cover picture for the region $[0,1]^2 \subset C$ and plate 1b for the region $[-0.88, -0.05] \times [-0.91, 0.06]$. The second picture has been selected by ACM/SIGGRAPH for inclusion in the SIGGRAPH' 83 Art Show.

There is a striking similarity between these pictures and figure 8, the binary decomposition for $R(z) = z^2$: It seems as if the dynamics in each connected component of $A(z_k)$, $k = 0,1,2$ is equivalent to the dynamics of $R(z) = z^2$ around 0. This is not so surprizing close to the fixed points z_k since in sufficiently small neighborhoods of z_k Newton's Method $N(z)$ essentially reduces to a quadratic map. But in fact, one can prove that there exists a conjugation between N and R in a more global sense:

Theorem

Let $N(z) = z^2\eta(z)$ with $\eta(0) = 1$ be a rational function, and let $A^*(0)$ be the maximal domain of normality of N which contains 0. Assume that 0 is the only critical point of N in $A^*(0)$, i.e., $N'(z) \neq 0$ for all





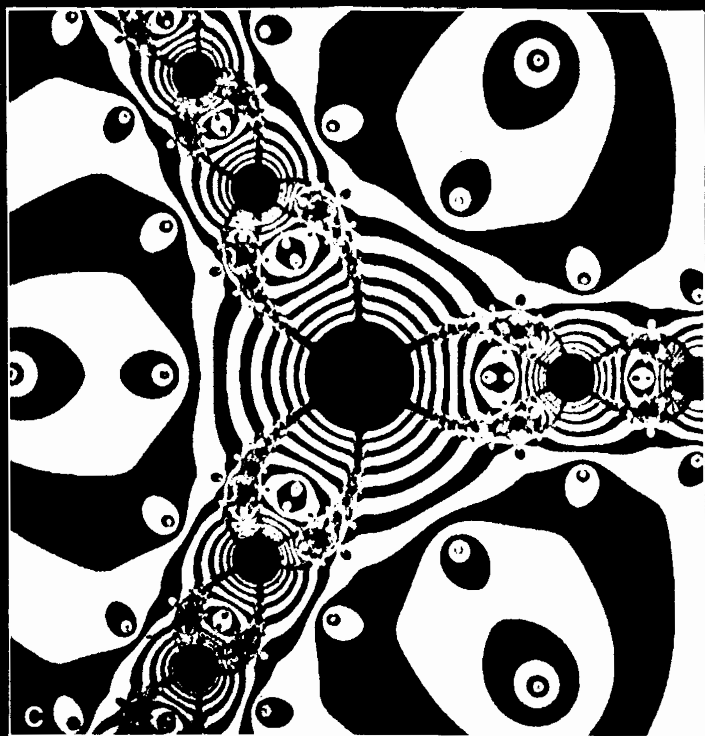


Figure 7. Study of the relaxed Newton method $N_\lambda(z) = z - \lambda p(z)/p'(z)$ for $p(z) = z^3 - 1$ in $[-1,1] \times [-1,1]$. We have $\lambda = 1.5$ in a), $\lambda = 1.0$ in b) and $\lambda = 0.5$ in c) and alternating colors for the level sets.

$z \in A^*(0) \setminus \{0\}$. Then there exists a conformal mapping

$$T: A^*(0) \to \{z : |z| < 1\}$$

such that

$$R = T \circ N \circ T^{-1}$$

where $R(z) = z^2$.

The existence of T is obtained locally near 0 by a method of successive approximations (Newton method), which in the limit satisfies (5). This is very much in the spirit of ideas developed by J. Moser [13] and H. Rüssman [16],[17] for the proof of the Twist Theorem. Then the extension of T to $A^*(0)$ is carried out in much the same fashion as in the construction of the binary decomposition of $A^*(0)$.†

The theorem may be applied to Newton's Method for $p(z) = z^3 - 1$ where, e.g., $z_0 = 1$ takes the role of 0. Thus, we see that Cayley's problem for $z^3 - 1$ finds at least a partial solution. However, in the step from $z^2 - 1$ to $z^3 - 1$ we loose in two ways: 1. The conjugation to $R(z) = z^2$ works only in connected components of the basins of attraction, and 2. the exact form of the conjugation T is not known, only the existence is verified.

Let us remark that the binary decomposition in the beginning was introduced only for aesthetical reasons. In a discussion with J. Hubbard we discovered an appropriate interpretation which eventually led to the theorem.

A last remark is in order. Naturally there are many more fascinating features of Julia sets. Among them are the topological structures of Julia sets J (when is J connected, totally disconnected, a Jordan curve?), their (Hausdorff) dimension (see [15]) and their relation to the KAM theory. An introduction into these aspects is given in [14]. For a survey we recommend

† Note added in proof: Using different methods the theorem had also been proved by Boettcher (1905) in the *Bulletin of the Kasan Mathematical Society*, vol. 14, p. 176.

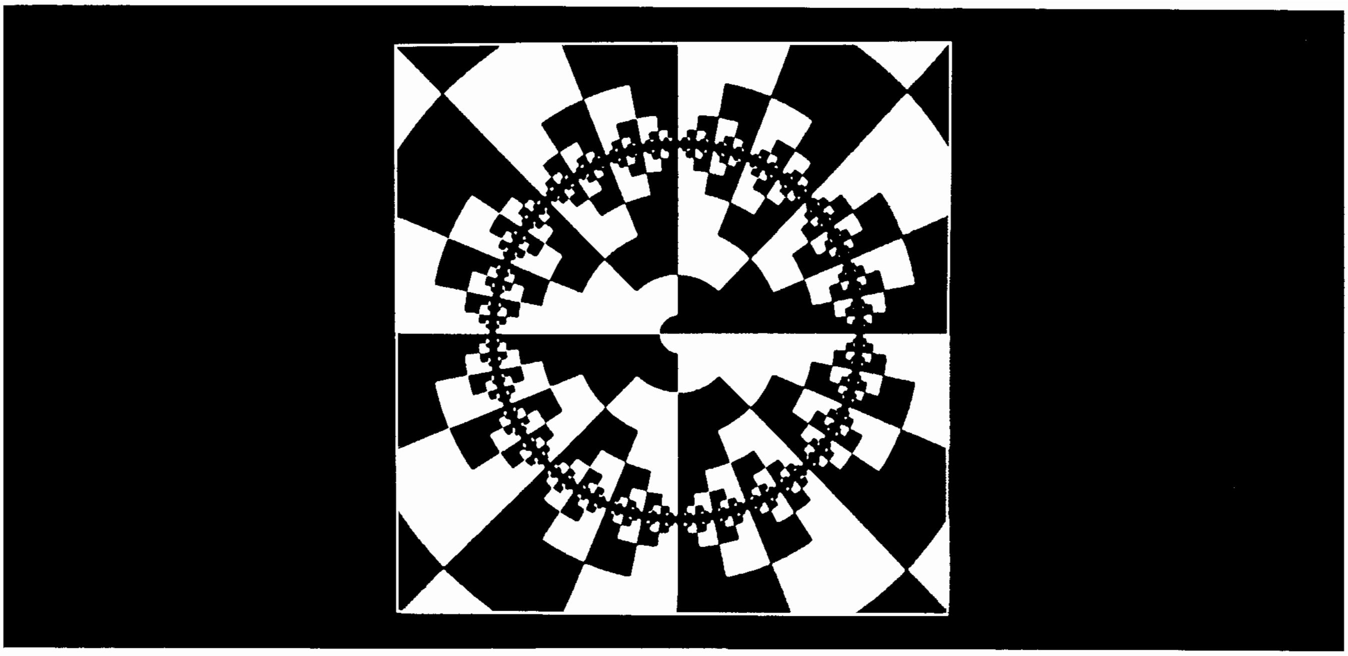


Figure 8. Binary decomposition of level sets of $R(z) = z^2$. The domain shown is $[-3/2,3/2] \times [-3/2,3/2]$.

the papers by H. Cremer [6], H. Brolin [2], J. Guckenheimer [9] and A. Douady [7], the latter one, explaining recent results of A. Douady, M. Herman, J. Hubbard, and D. Sullivan, clearly shows that the subject has returned to the attention of present mathematics after almost 70 years. This revival has certainly been promoted by the findings of Mandelbrot [12].

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