Fractal Images - From Mathematical Experiments to Phantastic Shapes : Dragon Flies, Scorpions and Imaginary Planets

Heinz-Otto Peitgen* and Dietmar Saupe

University of Bremen and University of California, Santa Cruz

0. Introduction

The goal of this lecture is to demonstrate how genuine mathematical research experiments open a door to a seemingly inexhaustible new reservoir of phantastic shapes and images. Their aesthetic appeal stems from structures which at the same time are beyond imagination and yet look extremely realistic. Being the result of well defined mathematical processes which depend on a few parameters animations of evolution or metamorphosis as well as continental drifts on imaginary planets, for example, are easy to obtain.

Besides selected stills we will show excerpts from several movies ("Experience Fractal Images", "Fly Lorenz"). Most of our material can be seen during the time of Siggraph '85 in the SF EXPLORATORIUM (Golden Gate Park) in a special exhibition: "Frontiers of Chaos - Computer Graphics Face Complex Dynamics". As an addendum to these notes we also refer to the brochure [PR] which will be part of the course notes package.

The experiments which we present here are part of a world wide mushrooming interest in complex dynamical systems. They deal with chaos and order and with their competition or coexistence. They show the transition from one to the another and how magni-

*) Lecture given as an invited course on SIGGRAPH '85 in San Francisco

ficently complex the transitional region generally is. And though they revel in the forms in these regions, they are also an attempt to understand the central question of how the structure of the boundaries depends on parameters. This leads us to new boundaries at another level - and reveals regularities that no one had suspected a few years ago.

The processes chosen here come from various physical or mathematical problems. They all have in common the competition of several centers for the domination of a plane. A simple boundary between territories is seldom the result of this contest. Usually, an unending filigreed entanglement and unceasing bargaining for even the smallest areas results.

It is the border region where the transition from one form of existence to another takes place: from order to disorder, magnetic to non-magnetic state, or however the regions which meet at the boundary are to be interpreted. The regions are more or less meshed at the boundary depending on the conditions characterizing the process in question. Occasionally, a third competitor profits from the dispute of two others to establish its area of influence. It can happen that one center dominates the entire plane - but there are still "boundaries" of its power, namely, isolated points which are not subjected to its attraction. Dissidents, so to say, who don't want to belong.

These pictures represent processes which are, of course, simplified idealizations of reality. They exaggerate certain aspects to make them clearer. For example, no real structure can be magnified repeatedly an infinite number of times and still look the same. The principle of self-similarity is nonetheless realized approximately in nature: in coastlines and riverbeds, in cloud formations and trees, in the turbulent flow of liquids and in the hierarchical organization of living systems. It was Benoit B. Mandelbrot who has opened our eyes to the "fractal geometry of nature".

Actually, the processes which yield such structures have been studied for a long time in mathematics and physics. They are simple feed-back processes in which the same operation is carried out repeatedly, the output of one iteration being the input for the next one:

The only requirement here is a nonlinear relation between input and output, i.e. the dynamical law $x_{n+1}=f(x_n)$ must be more than a simple proportionality, $x_{n+1}=kx_n$.

If we start this kind of iterative process with an arbitrarily chosen x_o , it will generate a sequence of values $x_o, x_1, x_2, \ldots$, whose behavior over long periods of time is our interest. Will the sequence tend toward a particular limit value X and come to rest there? Will it arrive at a cycle of values which is repeated over and over again? Or is the sequence erratic for all times, determined by the dynamic law and the initial value but nevertheless unpredictable?

Processes of this kind are found in all exact sciences. Indeed, the description of natural phenomena by differential equations, introduced by Sir Isaac Newton and Gottfried W. Leibnitz over 300 years ago, is based on a feedback principle: The dynamical law determines the location and velocity of a particle at one time instant from their values at the preceding instant. The motion of a particle is understood as the unfolding of a dynamical law. It is not essential here whether the process is discrete - that is, it takes place in steps - or continuous. Physicists like to think in terms of infinitely small, "infinitesimal" time-steps: natura non facit saltus. Biologists, on the other hand, often prefer to look at the changes from year to year or from generation to generation. Obviously, both views are possible, and the circumstances stimulate which description is appropriate.

Our notes are organized as follows: In the first paragraph we survey some elementary facts about phase portraits of maps. The second paragraph is devoted to Julia and Julia-like sets. Julia sets play a fundamental role in the theory of complex analysis. The third paragraph introduces into the theory of critical points and surveys the Mandelbrot set. The last paragraph collects a few special effects which are chosen to demonstrate the computer graphical potential of our structures.

1. Complex Boundaries as a Result of Competition

Elementary and at the same time incredible rich mathematical models for competition are obtained from phase portraits of C^1 maps

$$f : \mathbb{R}^n \to \mathbb{R}^n .$$

Let P be a fixed point of f, i.e. $f(P) = P$. Then P is said to be

repelling $\Leftrightarrow$ all eigenvalues of $f'(P)$ have modulus greater than one

attractive $\Leftrightarrow$ all eigenvalues of $f'(P)$ have modulus less than one

If P is an attractive fixed point of f then its basin of attraction $A(P)$ is the set of points

$$A(P) = \{x \in \mathbb{R}^n : f^k(x) \to P \quad \text{as} \quad k \to \infty\}$$
$$(f^k = f \circ f \circ \ldots \circ f \quad k\text{-times})$$

Note that f leaves $A(P)$ invariant. A periodic point P of f of period $p \in \mathbb{N}$ is a point such that $f^p(P) = P$ and $f^k(P) \neq P$, $1 \leqslant k < p$.

Any periodic point P creates a periodic cycle

$$\{P, f(P), \ldots, f^p(P)\} .$$

Such a cycle is said to be repelling (resp. attractive) provided

p is a repelling (resp. attractive) fixed point of the map f^p . A natural extension of the notion of a basin of attraction for an attractive cycle is obvious.

One of the important questions in dynamical systems theory is to characterize boundaries of basins of attraction. For example are they smooth manifolds or just topological manifolds or no manifold at all? What is their dimension (to be precise Hausdorff-dimension)? If f has several attractive cycles (i.e. there is competition between several centers) what can be said about the boundaries of their basins of attraction? For example if there are many attractors a natural guess would be that there are always two such that their basins of attraction share a common boundary. It should be noted that except for very particular examples (see 2.) nothing is known with regard to these questions, while it is obvious that they are extremely important, because many problems in science have to be modelled by discrete time processes.

A good way to bring light into these problems is to run computer graphical experiments, e.g. if $n = 2$.

(1.1) Approximation of Boundaries by Level Sets

Usually there is no explicit characterization of the boundary ∂A of a given basin of attraction $A(P)$. But there is a neat way to approximate it in a natural way. Points on the boundary ∂A cannot converge to P under iteration of the underlying map f . However, points in $A(P)$ arbitraryly close to the boundary eventually must converge to P , and, roughly speaking, the closer the initial point lies to the boundary the longer it takes the iteration to converge to P . Thus, we define a "target" set D containing P , e.g. a small disk centered at P , and approximate the basin $A(P)$ by the set A_k of points z which require at most k iteration to hit the target set D :

$$A_k = \{z \in \mathbb{C}, f^i(z) \in D \quad \text{for some} \quad i=0,\ldots,k\}$$

As k grows, the approximation will become better. The points which lie in A_k but not in A_{k-1} constitute a "level set of equal attraction" L_k :

$$L_k = A_k \setminus A_{k-1} \ .$$

In an obvious way we have that L_k approximates the boundary ∂A as k grows.

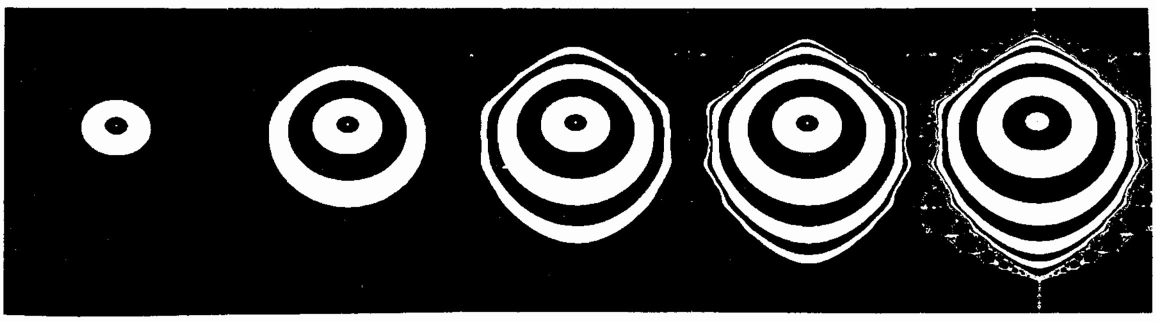

Fig. 1 Sequence of level sets around an attractive fixed point

(1.2) Binary decompositions and arbitrary targets

We are not restricted to disks when choosing target sets. For example we may choose a set containing P which has a fractal boundary. The fractal boundary of the target set carries over to the boundaries of all approximations A_k and all level sets L_k. The fractal boundary of A will be approximated by curves which are fractal themselves ... Other interesting results are obtained by choosing disks around P with respect to other norms, e.g.

$$\| (x,y) \|_p = (|x|^p + |y|^p)^{1/p}$$

with $p < 1$.

One can further subdivide the target sets into several portions thus subdividing all subsequent level sets into corresponding subsets. As an example we illustrate the outcome for a disk cut through the center into two parts yielding "binary decompositions" of level sets.

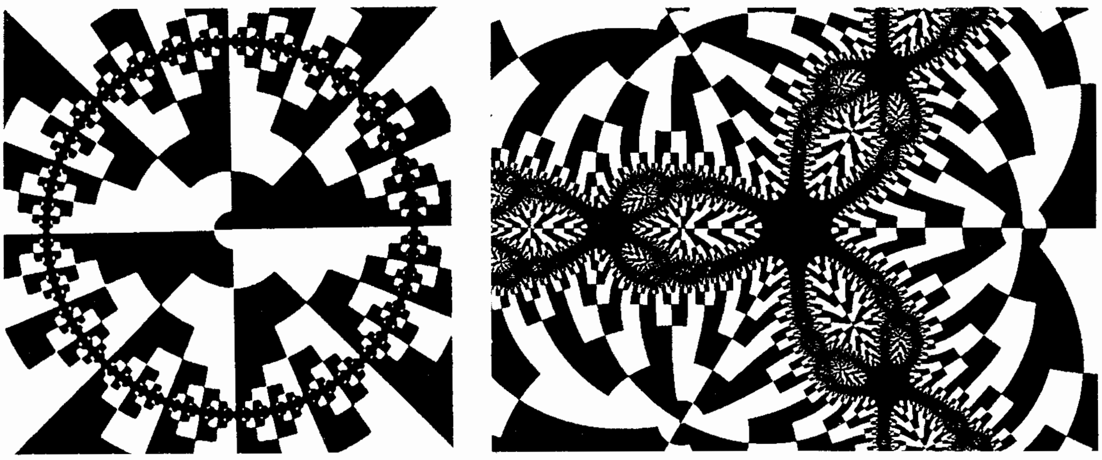

Fig. 2 Binary decomposition of level sets

2. Julia and Julia-like Sets

In this paragraph we discuss two classes of examples. The first one - nowadays called Julia sets after the pioneering work of G. Julia in the 1910's - is provided by rational maps R of the complex plane $\mathbb{C}$. Here is a small collection of the list of the numerous known results (see [B] for a recent survey):

Let $R: \mathbb{C} \to \mathbb{C}$ be a rational map of degree greater than 1, then the Julia set J_R is by definition the closure of all its repelling cycles.

(2.1) J_R is non-empty and closed

(2.2) If P is an attractive periodic point and $A(P)$ is the basin of its cycle, then

$$J_R = \partial A(P) .$$

(2.3) If P is a repelling periodic point, then J_R is the closure of

$$\{x \in \mathbb{C} : R^k(x) = P \text{ for some } k \in \mathbb{N}\} .$$

Note that (2.3) allows for a direct numerical (graphical) representation of J_R . Property (2.2) looks innocent but has striking consequences. For example, if R has several attractors then the boundaries of their basins are all identical! Fig. 3 shows the map $R(z)=(2z^3+\frac{1}{4}iz+\frac{5}{4})/(3z^2+\frac{1}{4}+\frac{1}{4}i)$

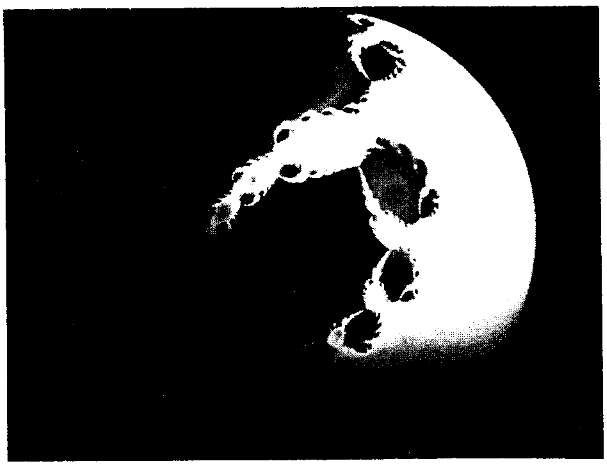

Fig. 3

on the Riemannian sphere S identified with $\mathbb{C} \cup \{\infty\}$.
Note that this map has 3 attractive fixed points and thus according to (2.2) each point on the boundary of their basins of attraction is a three-corner-point. It is a typical fractal of Hausdorff-dimension greater than 1. See also Maps 56,57 in [PR] .

How does one obtain "good" examples. A nice reservoir comes from a very traditional method in numerical analysis:

(2.4) Newton's Method

Given any map $f: \mathbb{R}^n \to \mathbb{R}^n$, $f \in C^1$, then one can solve

$$f(x) = 0$$

by the method

$$x_{n+1} = x_n - (Df(x_n)^{-1})f(x_n) =: N(x_n) ,$$

where $Df(x_n)$ denotes the derivative of f at x_n . Zeroes of f then give rise to attractive fixed points of N . For maps in $\mathbb{C}$ we modify the method

(2.5) $$N_{\varepsilon,r}(x) := x - r \frac{f(x)}{f'(x)+\varepsilon \cdot i} ,$$

with $r \in \mathbb{C}$, and $\varepsilon \in \mathbb{R}$.

The parameters ε, r allow continuous changes of the associated Julia sets. Fig. 4-6 show $f(x)=z^3-1$, $\varepsilon = 0$ and $r = 1.5$, $r = 1.0$, $r = 0.5$. Shown are level sets of the 3 basins for the roots of unity in alternating black and white.

Other interesting examples will be obtained by almost any rational map of $\mathbb{C}$ chosen at random.

Well studied (see [M]) is the family

(2.6) $$f_c(x) = x^2 + c , \qquad c \in \mathbb{C}$$

and the family (see [PR])

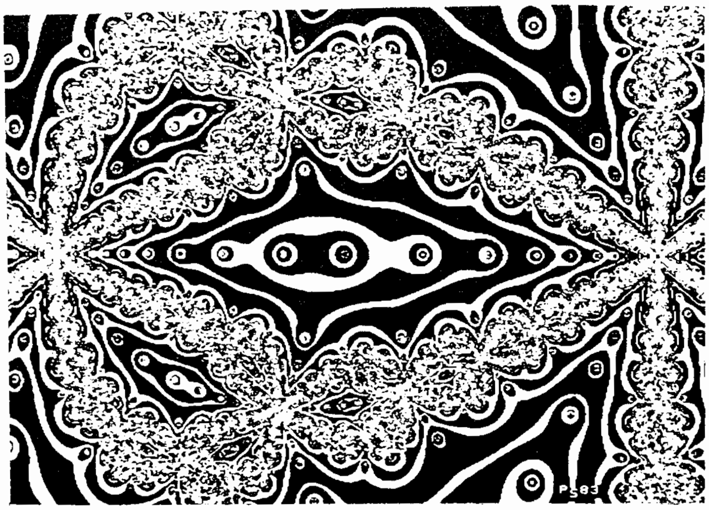

Fig. 4

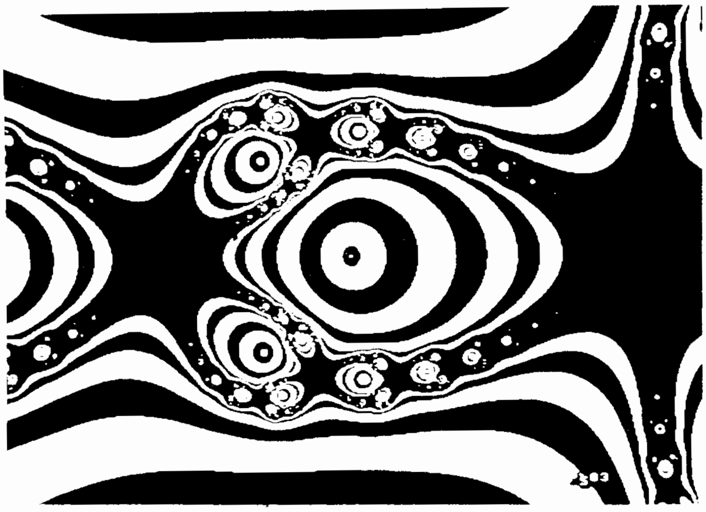

Fig. 5

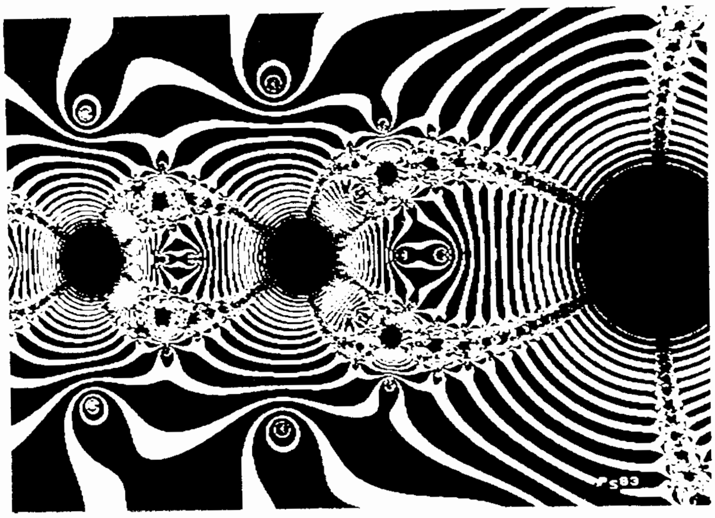

Fig. 6

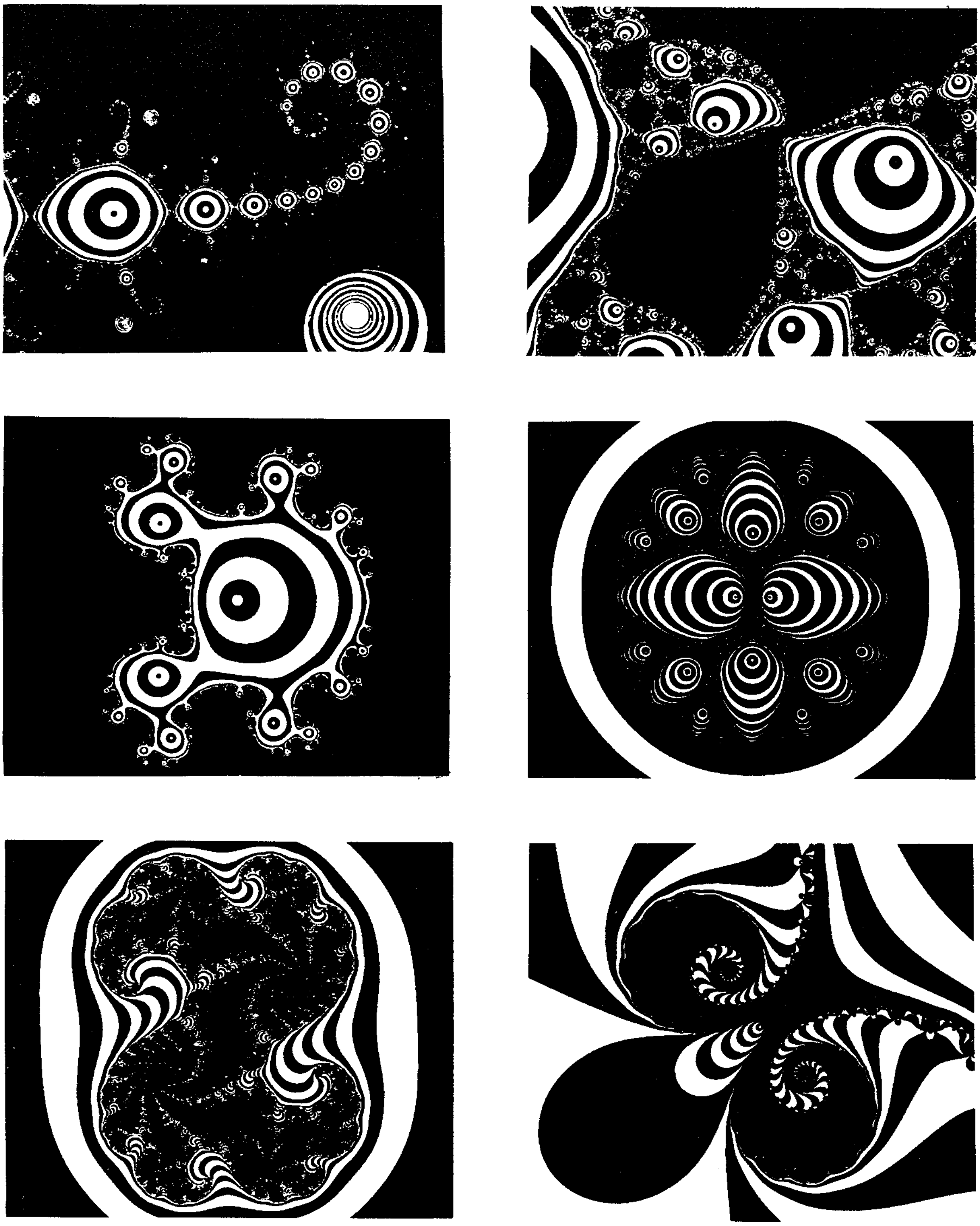

Fig. 7 Examples of competition in a lattice model for ferromagnetism. See also Maps 3-10 in [PR].

$$R_q(x) = \left(\frac{x^2+q-1}{2x+q-2}\right)^2 . \tag{2.7}$$

The latter one plays a central role in models of phase transitions in statistical mechanics. It is the renormalization transformation of a certain lattice model for ferromagnetism (figure 7).

While the situation is more or less completely understood by now for rational maps of $\mathbb{C}$ almost nothing is known for general maps of $\mathbb{R}^n$. Our interest has been for some time to understand discretizations of non-linear boundary value problems

$$\begin{aligned} u'' + \lambda f(u) &= 0 \\ u(0) = 0 &= u(1) \end{aligned} \tag{2.8}$$

where f is a non-linear function and λ is some external control parameter. Discretizing (2.8) by central differences on two internal meshpoints $1/3,\ 2/3$ yields the system

$$Ax - \frac{1}{9}\lambda F(x) = 0 \tag{2.9}$$

where A is the matrix $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ and $x = (x_1,x_2)^T$ and $F(x) = (f(x_1),\ f(x_2))^T$.

Zeroes of $G(\lambda,x) = 0$ are then numerical approximations of solutions of (2.8) where

$$G(\lambda,x) = Ax - \frac{1}{9}\lambda\ F(x) .$$

Newton's method defined by

$$H(\lambda,h,x) = x - h(DG(\lambda,x)^{-1})G(\lambda,x) \tag{2.10}$$

again is a prominent method to solve (2.9). For certain theoretical reasons we have studied in great detail the

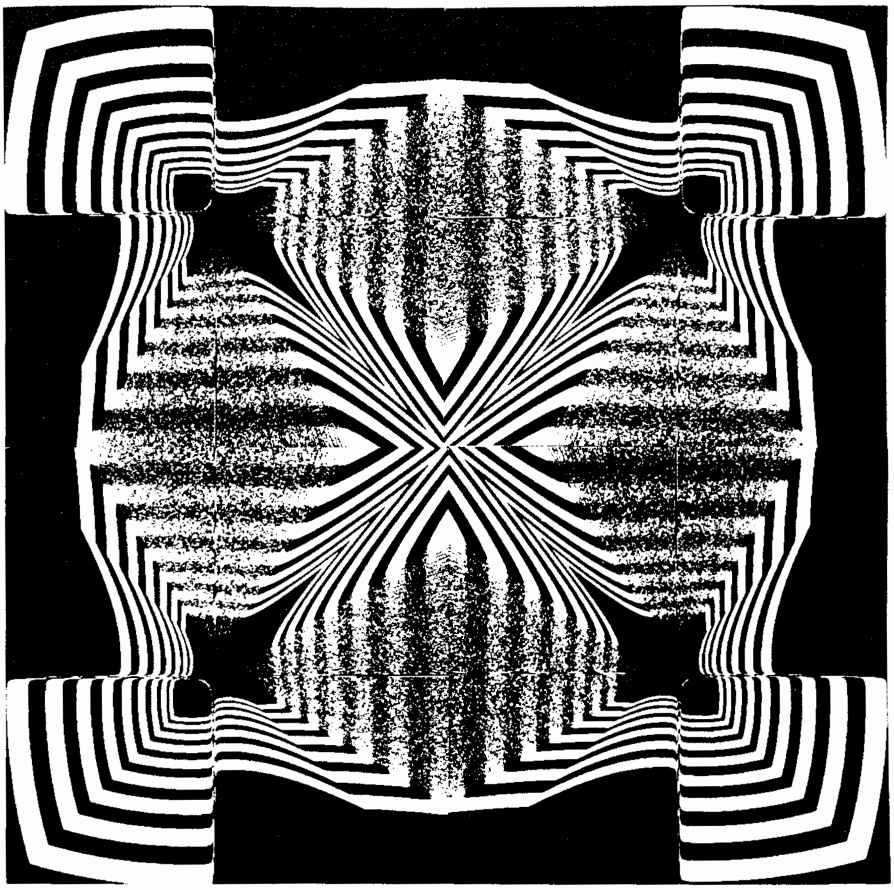

Fig. 8 Phase portrait of (2.10) with $h = 0.2$, $\lambda = 28.8$.

Fig. 9 Same phase portrait with $h = 1.7$ (left), $h = 1.8$ (right) .

function

$$f(s) = s - s^2 .$$

The parameter h - sometimes called relaxation parameter - can be viewed as a discretization constant itself. Indeed, (2.10) is just the Euler method for the initial value problem

$$(2.11) \qquad \dot{x} = - (DG(\lambda,x)^{-1})G(\lambda,x)$$

The parameters λ and h play an important role in understanding the phase portrait of map (2.10).

One of the central questions in investigating (2.10) was to characterize the boundaries of basins of attraction. Are they always identical - as for rational maps of $\mathbb{C}$ - or may different basins have different boundaries. It turns out that the latter is the case and more specific results were only possible after the above computer graphical experiments were done.

3. Mandelbrot and Mandelbrot-like Sets

The famous Mandelbrot set (Fig.12) plays a central role in the discussion of a metamorphosis of boundaries. It is attached to the family (2.6):

$$f_c(x) = x^2 + c , \quad c \in \mathbb{C} .$$

Note that $f_c(\infty) = \infty$ for all $c \in \mathbb{C}$ is an attractive fixed point. Thus, there is always the basin $A(\infty)$. For some values of c - e.g., if $|c| \ll 1$ then f_c has an attractive fixed point z_a - there is a second attractive cycle which establishes a second basin $A(z_a)$. It is less obvious but true that for $|c| << 1$ f_c has only one attractive cycle and that is of course the point ∞ . Even then - according to (2.1) and (2.2) - we have $\partial A(\infty) \neq \emptyset$. Now is a deep fact that $J_c = \partial A(\infty)$, the Julia set of f_c , is either connected or it is a Cantor set provided $c \in M$ or $c \notin M$, where M is the Mandelbrot set:

$$(3.1) \qquad M = \{ c \in \mathbb{C} : f_c^k(0) \not\to \infty \text{ as } k \to \infty \}$$

The point 0 is a critical point for f_c and the above result is an example of the critical point theory for rational maps of $\mathbb{C}$ (see [B]). Thus, precisely at the boundary of M the Julia set J_c undergoes a dramatic crisis, i.e. as one passes from the inside of M to its exterior the associated Julia sets change from a connected set into a totally disconnected object.

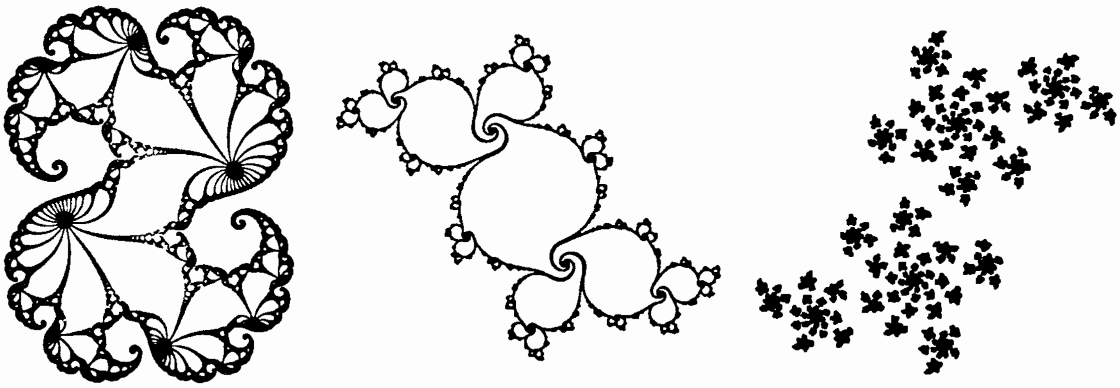

Fig. 10 3 examples of Julia sets the first two being connected and the last being a Cantor set

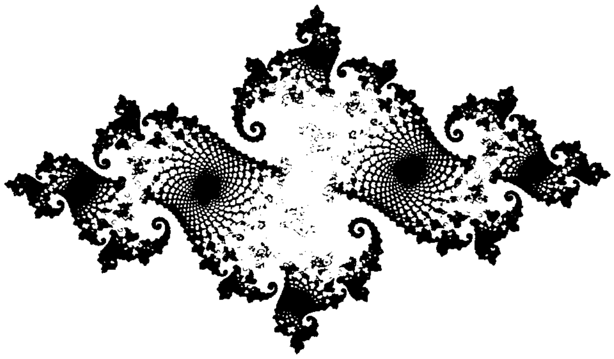

Fig. 11 Another example of a Cantor set

Therefore the boundary of M is of particular importance and interest. How can it be made visible? Here is a good way to think about M : Think of M as a piece of metal being charged electrostatically. Now measure and draw the potential of M in the exterior of M by drawing equipotential lines. Thus, the "picture of the potential" will give a reasonable approximation of ∂M near it. The method to compute the potential is fortunately very cheap due to a wonderful result of A. Douady and J. H. Hubbard (1982). The potential, it turns out, is obtained by just measuring the "escape time to ∞ " of the critical point 0 for $c \notin M$ (see (3.1)). The level sets in figure 12 are obtained in that way.

4. Special Effects

With regard to the computer graphical impact of the above structures we want to conclude with several remarks:

(4.1) Parameter Dependance

All structures depend in natural ways on parameters which create an enormous potential for phantastic animations. For example a metamorphosis of the "dragon fly"

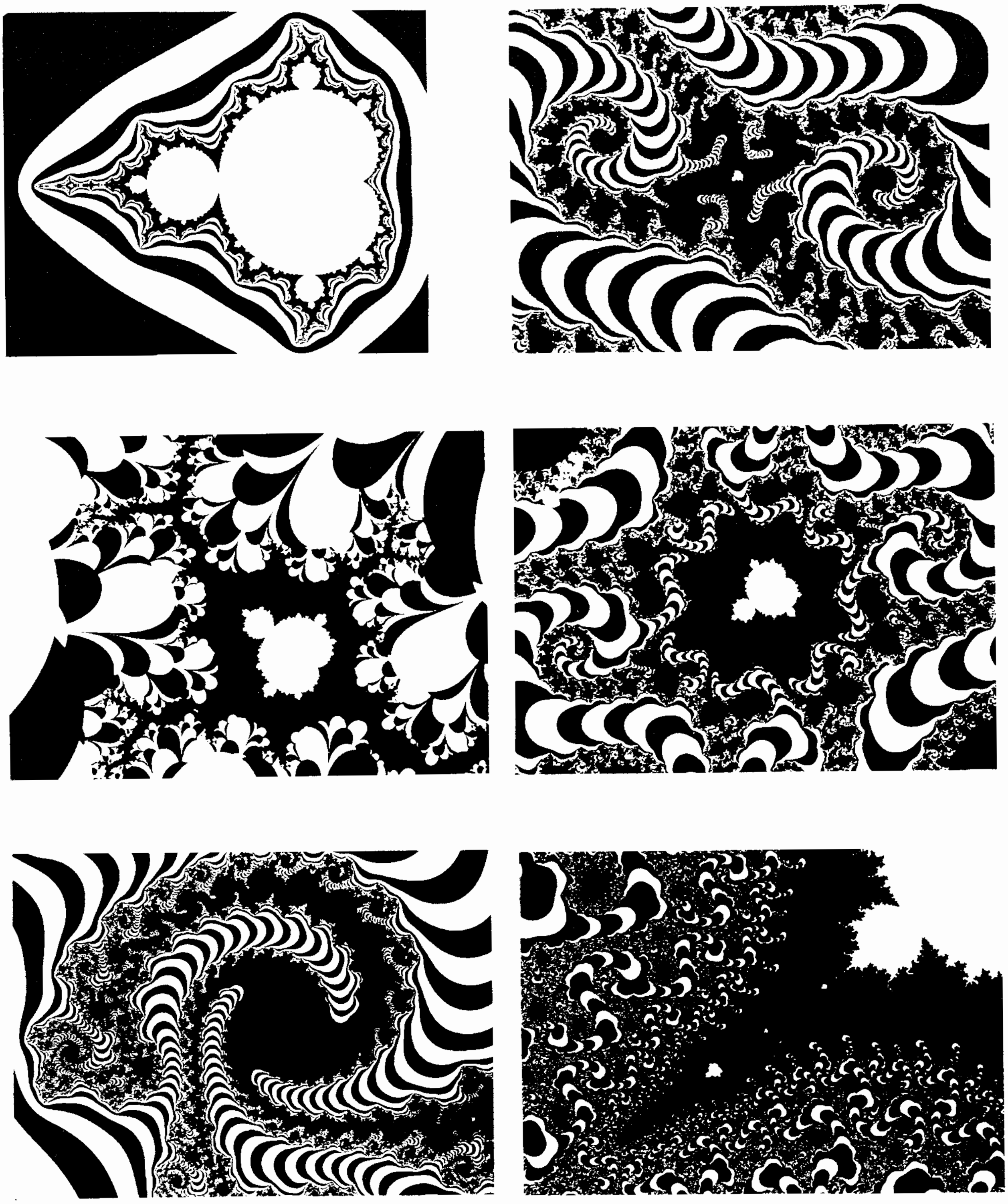

Fig. 12 The Mandelbrot set with its equipotential lines and some enlargements at its boundary. See also Maps 22-44 in [PR].

into "batman" (figure 9) is easily arranged by just changing the step size h from 1.7 to 1.8 (λ= 28.8). Similarily all structures in Fig. 10, 11 can be easily transformed into each other by just changing c in f_c continuously. This gives rise to all kinds of evolution - and metamorphosis - effects. Applying these ideas one could quite easily animate the evolution of a continental drift starting with the imaginary planet of figure 3 and ending with the one below

Fig. 13 Transformed imaginary planet

by just choosing the appropriate family of maps $g_t(x)$, $0 \leqslant t \leqslant 1$, together with Newton's method .

(4.2) Internal Substructures

Most of the above images are in fact images composed by an enormous number of level sets. Thus, our images are set up by well-defined substructures. This together with coloring and shading gives rise to a nice toolkit to play with:

Provided that a look-up-table for 256 or more colors is available, one should link each level set to an entry of the look-up-table. By shifting colors through parts of the table completely different appearences of the displayed object may occur (see Map 50-54 in [PR]) . Color map animation may then be used to continuously transform one image into the other. If the planar image is projected onto a sphere as in figure 3 one can employ standard shading techniques to obtain additional effects.

All our color pictures (Map 1-70, [PR]) have been composed in the spirit of the above ideas, i.e. rather than using a coloring algorithm geared to the mathematical background of the images, we used an interactive coloring program that allows a variety of color map manipulations to create different effects.

(4.3) "Fly Lorenz"

This excerpt from the 13min 16 mm movie under the same title (Authors: H.O. Peitgen and H. Jürgens, Fly Lorenz, Institut für den Wissenschaftlichen Film, 1984. Order film from: Int. Film Bureau Inc., 332 South Michigan Av., Chicago, Illinois 60604) shows a 3D animation of the famous Lorenz attractor which is the first historical example of a strange attractor for continuous differential equations:

$$\dot{x} = -\sigma x + \sigma y$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy + bz$$

$r = 28$, $\sigma = 10$, $b = 8/3$.

References:

[B] P. Blanchard: Complex analytic dynamics on the Riemann sphere, Bull. AMS 11, 85-141, 1984.

[M] B. Mandelbrot: The fractal geometry of nature, Freeman, San Francisco 1982.

[PR] H.-O. Peitgen, P. H. Richter: Frontiers of chaos, Bremen, 1985.