Topological Perturbations in the Numerical Study of Nonlinear Eigenvalue- and Bifurcation Problems

BY

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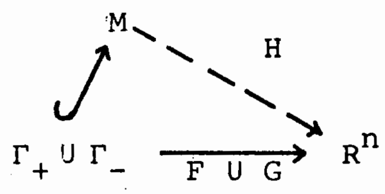
1. INTRODUCTION

The aim of this paper is to continue and extend the discussion of a new device in the context of simplicial and continuation methods which was introduced in [P-P] and to relate this device with a suitable interpretation of an idea due to M.M. Jeppson [J,A-J]. As an application their usefulness is demonstrated in some selected numerical problems: an elliptic boundary value problem with infinitely many solutions, a bifurcation problem with non-differentiable nonlinearity, a periodicity problem given by a difference differential equation which is conjectured to have chaotic behaviour. For an efficient implementation of our topological perturbation techniques it has been most important to introduce and use a new concept of triangulation. This concept which in a sense can be understood as a "virtual" triangulation introduces a large amount of flexibility into simplicial path following algorithms, e.g. the mesh size can be modified and still the triangulation process is as simple as Kuhn's triangulation. Among other consequences the design of a new and very effective acceleration technique will be discussed.

The setting of topological perturbation is given by the following choices: Let $M \subset R^{n+1}$ be a triangulable subset of homogeneous dimension $(n+1)$ and let $F:M \to R^n$ be continuous. We study the problem $F^{-1}(O)$. Let $\Gamma_+, \Gamma_- \subset M$ be disjoint sets which are the closure of open sets in R^{n+1}. Choose a continuous map $G: \Gamma_- \to R^n$ and an extension $H: M \to R^n$ of F and G (i.e. a map which makes the following diagram commutative).



Then H fixes a continuous perturbation of F outside of Γ_+. Note that since R^n is an AR (absclute retract) [Du] one has an extension for any pair Γ_+, Γ_- and F,G. The aim of topological perturbations is here to tackle two problems:

- generate continua in M which are solutions of $H^{-1}(0)$ and which connect certain solutions in $F^{-1}(0)$ and which thereby make these solutions accessible with the aid of path following algorithms;
- accelerate simplicial path following algorithms.

The technique $(F,G,H,\Gamma_+,\Gamma_-,M)$ will have as typical choices:

- Γ_+ and Γ_- half-spaces or cubes in R^{n+1};
- $G^{-1}(0) = \emptyset$;
- $G^{-1}(0) \subsetneqq F^{-1}(0)$;
- $G^{-1}(0) \cong F^{-1}(0)$ ($\cong$ isomorphic).

Much of [P-P] is devoted to the powerful interplay of the concepts of completely labelled simplices in the context of simplicial algorithms and the Brouwer degree of a mapping. In fact [P-P] shows how a suitable interpretation of Brouwer degree is the key to understand the design of new devices for the numerical study of nonlinear eigenvalue and bifurcation problems. Since we need this illuminating bridge from topology to numerical analysis in an explicit form anyway we describe it here in short from a different point of view than in [P-P,G]:

Let U be open and bounded in R^n (n-dim. euclidean space) and let $f: (\mathrm{cl}U, \partial U) \to (R^n, R^n \setminus \{0\})$ be continuous. Then the Brouwer degree $\deg(f,U,\mathbf{0}) \in Z$ (Z denotes the integers) is defined which is a measure for the number of zeroes of f in U (cf. [A_1,A_2,B,D,E-F,K,M]). Now let T be a triangulation of R^n and let f_T denote the natural PL-approximation to f determined by T (PL = piecewise linear). Then f_T will be close to f provided the mesh size of T is fine. Thus $f_T:(clU,\partial U) \to (R^n, R^n \setminus \{0\})$ and then the continuity of $\deg(\cdot,U,0)$ will imply

$$\deg(f,U,0) = \deg(f_T,U,0) \; . \tag{1.1}$$

If 0 was a regular value for f_T (i.e. $f_T:(\sigma,\partial\sigma) \to (R^n, R^n \setminus \{0\})$ for all $\sigma \in T_n$, $\sigma \subset U$; T_k denotes the k-skeleton of T) and T sufficiently fine then we could proceed in the spirit of Brouwer's definition and use the additivity property to write

$$\deg(f_T,U,0) = \sum_{\substack{\sigma\in T_n \\ \sigma\subset U}} \deg(f_T,\mathrm{int}\sigma,0).$$

However, it will be crucial - and this is the only side step from Brouwer's definition - to obtain regularity by considering $\deg(f_T,U,\bar{\varepsilon})$ where $\bar{\varepsilon} \in R^n$ is sufficiently close to 0 and of the special form $\bar{\varepsilon} = (\varepsilon,\varepsilon^2,\ldots,\varepsilon^n)$, $\varepsilon > 0$. Of course, choosing any $\bar{\varepsilon} \in R^n$ sufficiently small one would have that f_T: $(\sigma,\partial\sigma) \to (R^n, R^n \setminus \{\bar{\varepsilon}\})$ for any $\sigma \in T_n$, $\sigma \subset U$ and therefore we could conclude from the continuity of $\deg(f,U,\cdot)$ and the additivity property

$$\deg(f,U,0) = \deg(f_T,U,\bar{\varepsilon}) = \sum_{\substack{\sigma\in T_n \\ \sigma\subset U}} \deg(f_T,\mathrm{int}\sigma,\bar{\varepsilon}). \tag{1.2}$$

We emphasize the special choice for $\bar{\varepsilon}$ for the following reasons: Let γ: $[0,\infty) \to R^n$ be the curve $\gamma(\varepsilon) = (\varepsilon,\varepsilon^2,\ldots,\varepsilon^n)$ and let $H \subset R^n$ be any hyperplane. Then one has that $\mathrm{Im}\gamma \cap H$ is a finite set and therefore we find $\varepsilon_0 > 0$ such that $\mathrm{Im}\gamma \cap H = \emptyset$ for all $0 < \varepsilon \le \varepsilon_0$. Now oberserve that $T_U := \{\sigma \in T_n: \sigma \subset U\}$ is finite and each $\partial\sigma$ is determined by hyperplanes. Hence, we can assume that $\bar{\varepsilon}$ in (1.2) is of the form $\bar{\varepsilon} = (\varepsilon,\varepsilon^2,\ldots,\varepsilon^n)$ where $0 < \varepsilon \le \varepsilon_0$ for some ε_0. We now define

$$S_f^T(U) := \{\sigma\in T_U: \bar{\varepsilon}\in f_T(\sigma) \quad \text{for any} \quad 0 \le \varepsilon \le \varepsilon_0\} \tag{1.3}$$

$\sigma \in S_f^T(U)$ is called "completely labelled" and the summation in (1.2) can obviously be restricted to $S_f^T(U)$. Moreover, if $f_{T|\sigma} = A_\sigma + b_\sigma$, where A_σ is linear and $b_\sigma \in R^n$, then A_σ is non-singular for any $\sigma \in S_f^T(U)$ and we can define

$$or(\sigma) := \text{sign}\ \det A_\sigma \quad \text{(Orientation of } \sigma) \quad . \tag{1.4}$$

Finally, since $\deg(f_T, \text{int}\sigma, 0) = \text{sign}\ \det A_\sigma$ according to the theorem of Leray and Schauder, we obtain the formula:

$$\deg(f,U,0) = \sum_{\sigma \in S_f^T(U)} or(\sigma) . \tag{1.5}$$

The beauty of (1.3) is revealed by stating the fact that $\sigma \in S_f^T(U)$ if and only if Λ^{-1} is lexico-graphically positive:

$$\Lambda := \begin{pmatrix} 1 & \dots & 1 \\ f(a_o) & \dots & f(a_n) \end{pmatrix}, \sigma = \text{convex hull of } \{a_o, \dots, a_n\} . \tag{1.6}$$

This provides an easy way for numerical computation.

It would not be an exciting remark to say that (1.5) can be used as a definition for Brouwer degree. However, it seems that this observation becomes very valuable once one notices that all basic properties of degree theory (solution, excision, additivity and normalization property and most important also the homotopy property) can be proved without ever leaving the settings of triangulations, refinements and completely labelled simplices. Especially, the homotopy invariance which in any other definition of Brouwer degree is the hardest to get is here an easy and algorithmic consequence of the following principle which makes completely labelled simplices so valuable for numerical analysis (cf. [A-G_1,E,E-Sc,J-S,P-P]). In [P-P] formula (1.5) is exploited for a constructive approach to the Leray-Schauder Continuation Method which is essentially the homotopy invariance of degree.

<u>LEMMA 1.7.</u> <u>Let</u> $M \subset R^n \times [a,b]$ <u>be a triangulable subset of homogeneous dimension</u> $n+1$. <u>Let</u> $F: M \to R^n$ <u>be continuous and let</u> T <u>be a triangulation of</u> M. <u>Assume that</u> $\sigma_o \in T_n$ <u>(a co-dimension-one simplex) is completely labelled. Then we have</u>

(i) ALGORITHM

σ_o determines a unique chain

$$ch_F(\sigma_o) = (\ldots,\sigma_{-1},\sigma_o,\sigma_1,\ldots)$$

of completely labelled simplices in T_n and this chain carries a PL-manifold of dimension one denoted by $m_F(\sigma_o)$ and $m_F(\sigma_o)$ is a component of $F_T^{-1}(\bar{\varepsilon})$ being a collection of manifolds, $(\bar{\varepsilon} = (\varepsilon,\varepsilon^2,\ldots,\varepsilon^n)$, ε small) denoted by M_F.

(ii) CLASSIFICATION

$m_F(\sigma_o)$ is homeomorphic with S^1 (unit one sphere)

and $m_F(\sigma_o) \cap \partial M = \emptyset$, or

is homeomorphic with $[0,1]$, $(0,1)$ or $[0,1)$

and $m_F(\sigma_o) \cap \partial M = \partial m_F(\sigma_o)$.

(iii) ORIENTATION

If $\sigma_o \subset R^n \times \{a\}$ and $\sigma_s \in ch_F(\sigma_o)$ with $\sigma_s \subset R^n \times \{a\} \cup R^n \times \{b\}$ then

$or(\sigma_o) = or(\sigma_s)$ provided $\sigma_s \subset R^n \times \{b\}$

$or(\sigma_o) = -or(\sigma_s)$ provided $\sigma_s \subset R^n \times \{a\}$

(iv) DEGREE, LERAY-SCHAUDER CONTINUATION METHOD

If M is bounded and $F:(M,\partial M) \to (R^n, R^n \setminus \{0\})$ and T is sufficiently fine then we have

$$\sum_{\sigma \in S_{F_a}^T(M_a)} or(\sigma) = \sum_{\sigma \in S_{F_b}^T(M_b)} or(\sigma)$$

$$\| \qquad\qquad \|$$

$$\deg(F_a, \mathrm{int} M_a, 0) \qquad \deg(F_b, \mathrm{int} M_b, 0)$$

$(F_a := F(\cdot,a)$, $M_c := \{x \in R^n: (x,c) \in M\})$.

A negation of (1.7. iv) can be interpreted to be equivalent with the global bifurcation result due to P.H.Rabinowitz [R] and this has been made precise in [P-P]:

THEOREM 1.8. Let $F\colon R^{n+1} \to R^n$ be continuous and such that $F(0,\lambda) = 0$ for all $\lambda \in R$ (trivial solutions). Let $[\lambda_1,\lambda_2]$ be an interval and let U_i be open neighborhood of zero in $R^n \times \{\lambda_i\}$, $i=1,2$, such that

$$F^{-1}(0) \cap \mathrm{cl}U_i = \{0\}, \quad i=1,2.$$

Let $\Omega \subset R^n \times R$ be a "proper connection" for U_1 and U_2 and assume that

$$\deg(F(\cdot,\lambda_1),U_1,0) \neq \deg(F(\cdot,\lambda_2),U_2,0).$$

Then there exists $\delta_0 > 0$ such that any triangulation T of a neighborhood of Ω for which $\mathrm{mesh}(T) \leq \delta_0$ and which is compatible with $R^n \times [\lambda_1,\lambda_2]$ has the following property:

There exists at least one completely labelled simplex of dimension n $\sigma^n \in T_n, \sigma^n \subset (U_1 \cup U_2)$, such that $ch_F(\sigma^n)$ leaves Ω through $\partial\Omega\setminus(U_1 \cup U_2)$. This means that $M_F = F_T^{-1}(\bar{\varepsilon})$ ($\bar{\varepsilon} = (\varepsilon,\varepsilon^2,\ldots,\varepsilon^n)$, ε small) contains a component $m_F(\sigma^n)$ which is a PL-1-manifold and which has the property

$$m_F(\sigma^n) \cap (U_1 \cup U_2) \neq \emptyset \quad \text{and}$$

$$m_F(\sigma^n) \cap \partial\Omega\setminus(U_1 \cup U_2) \neq \emptyset .$$

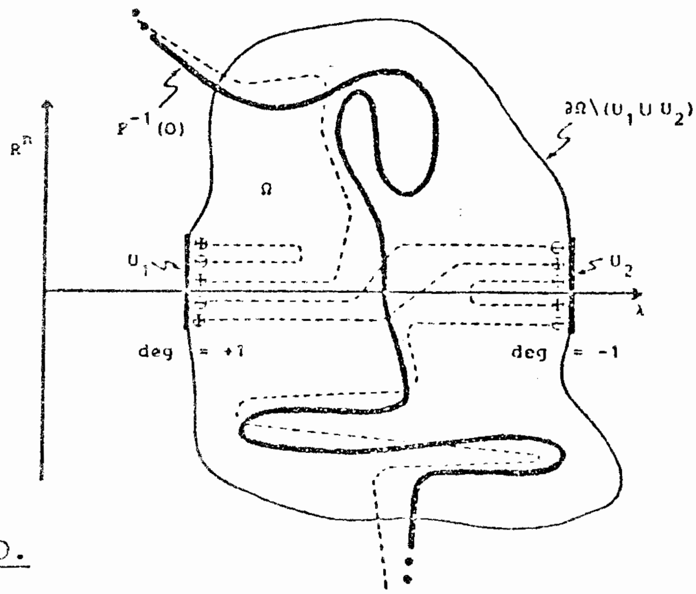


Figure 0.

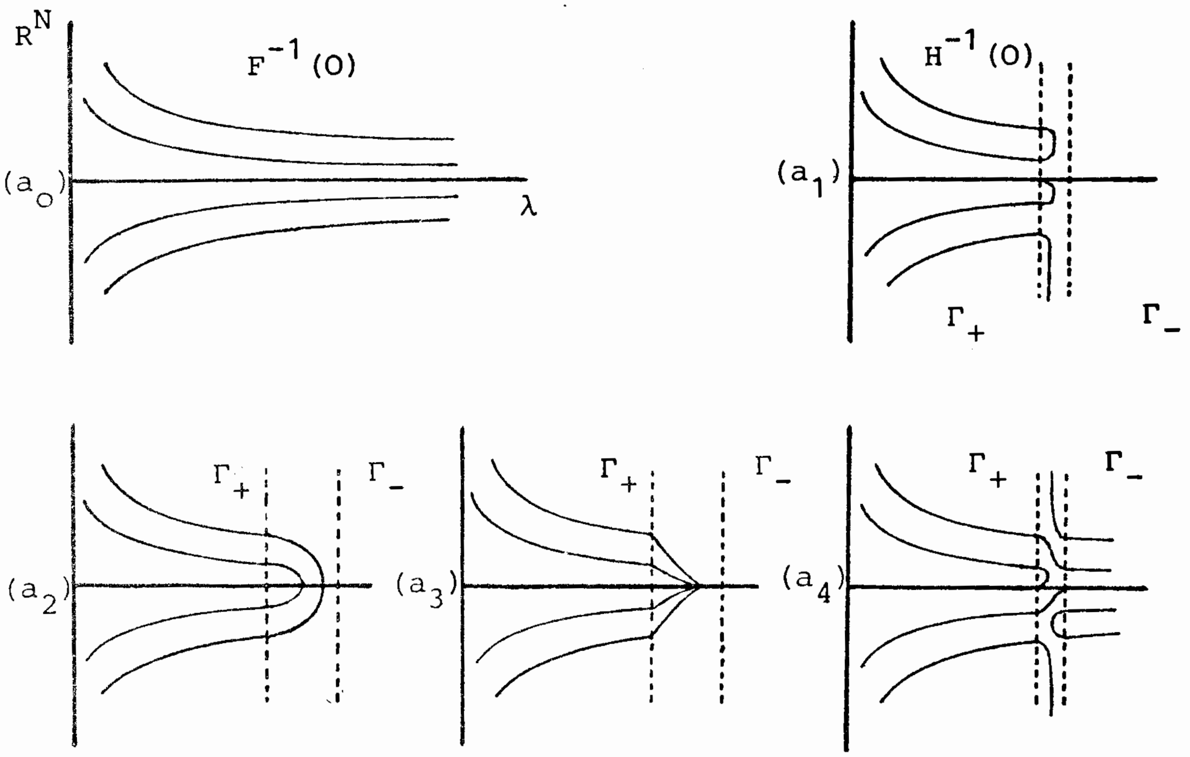
$\left\{\begin{matrix}\oplus\\ \ominus\end{matrix}\right\}$:= $\left\{\begin{matrix}\text{positively}\\ \text{negatively}\end{matrix}\right\}$ oriented compl. labeled simplices

------- := chains of compl. labeled simplices

(Ω is a proper connection for U_1 and U_2 provided $\Omega \cap \{0\} \times R = \{0\} \times [\lambda_1, \lambda_2]$ and Ω is connected and Ω has a decomposition $\Omega = \Omega^t \cup \Omega^n$ where Ω^t is open and bounded in $R^n \times [\lambda_1, \lambda_2]$, Ω^n is open and bounded in $R^n \times R$ and such that $\Omega^t_{\lambda_1} = U_1$, $\Omega^t_{\lambda_2} = U_2$ and $cl\Omega^t_{\lambda_1} \cap cl\Omega^n_{\lambda_1} = \emptyset = cl\Omega^t_{\lambda_2} \cap cl\Omega^n_{\lambda_2}$)

2. TOPOLOGICAL PERTURBATIONS I

In the following we study the problem $F(x,\lambda) = 0$ and assume for simplicity that F is defined on R^{n+1} rather than on a triangulable subset M of homogeneous dimension $(n+1)$. We note however that this case can be obtained in an obvious way from the following. Let Γ_+ and Γ_- be disjoint sets which are the closure of open sets in R^{n+1} and let $G: \Gamma_- \to R^n$ and let $H: R^{n+1} \to R^n$ be an extension of F and G. Here we focus on such perturbations H which generate continua in $H^{-1}(0)$ which provide a numerical access in the sense of continuation or simplicial methods to $F^{-1}(0) \cap \Gamma_+$. Figure 1 provides a geometric idea of two types of problems (problems with disjoint continua in $F^{-1}(0)$ and problems with bifurcation in $F^{-1}(0)$) and their perturbations:



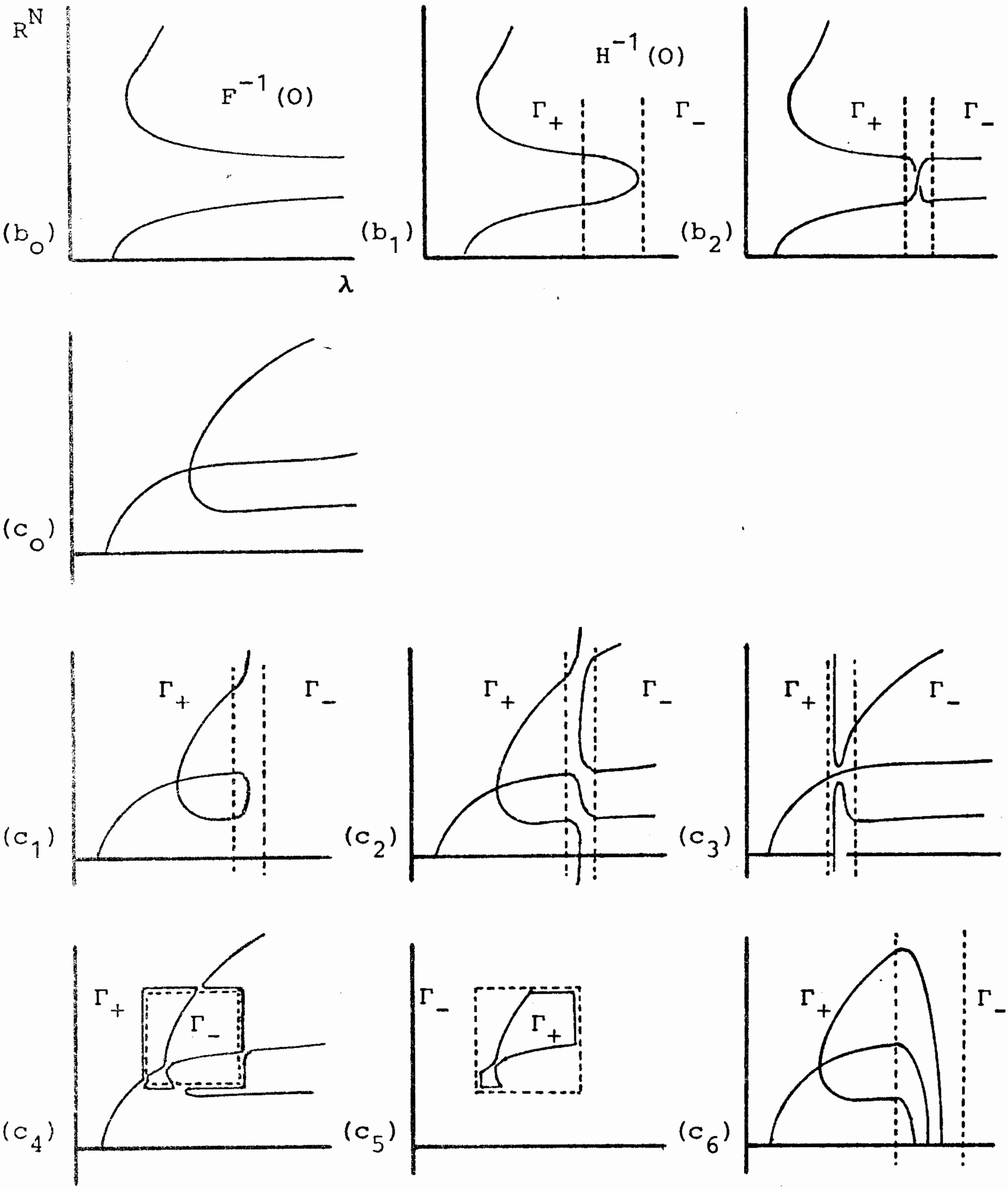


Figure 1.

Before we go into the details of definitions and abstract results we discuss some instructive examples. We consider as a model problem the nonlinear boundary value problem

$$\begin{cases} -u'' = \lambda f(u), \quad \lambda \in R, \quad f\colon [0,1] \to R \\ u(0) = u(1) = 0 \end{cases} \qquad (2.1)$$

Using central difference approximations for (2.1) on a uniform grid of meshsize h and N internal meshpoints in $[0,1]$ we obtain a problem which approximates (2.1) (cf. [Ke,S])

$$Ax = \lambda f(x) . \tag{2.2}$$

The associated zero-problem will be denoted by

$$F(x,\lambda) = Ax - \lambda f(x). \tag{2.3}$$

In the following we will discuss typical choices:

$$\Gamma_- = R^N \times [\lambda_*,\infty)$$

$$G(x,\lambda) = c \in R^N \setminus \{0\}$$

$$G(x,\lambda) = -\mathrm{Id}x$$

$$G(x,\lambda) = -Ax$$

$$G(x,\lambda) = Ax-\lambda g(x)$$

$$G(x,\lambda) = P \circ F(x,\lambda)$$

($Ax = \lambda g(x)$ will correspond to a problem $-u'' = \lambda g(u)$ and $u(0) = u(1) = 0$; $P \in GL_- = \{L: R^N \to R^N: L$ is a linear isomorphism of $\det(L) < 0\}$).

We will make an essential use of the following elementary lemmata.

<u>LEMMA 2.4.</u> ($G = -\mathrm{Id}$)

<u>Let</u> $f(x) = x^{2n+1}$ <u>and let</u> $|x| = \max_{1\le i\le N} |x_i|$. <u>Define</u> $h(x,t) := (1-t)(Ax-f(x)) - tx$ <u>for</u> $0 \le t \le 1$.

<u>Then one has the following properties for</u> $h^{-1}(0)$:

(i) <u>There exists</u> $r > 0$ <u>such that</u> $h^{-1}(0) \cap \partial B_r x[0,1] = \emptyset$ $B_r = \{x \in R^N: |x| < r\}$.

(ii) $h^{-1}(0) \cap B_r \times \{1\} = \{(0,1)\}$.

(iii) $h^{-1}(0)$ <u>contains precisely</u> N <u>bifurcation points</u> $t_1,\dots,t_N$ <u>with</u> $t_i \in (0,1)$ <u>and</u> $t_i = m_i(1+m_i)^{-1}$, <u>where the</u> m_i <u>denote the</u> N <u>distinct eigenvalues of</u> A. <u>Furthermore</u>, $h^{-1}(0) \subset B_r \times [0,1]$ for some $r > 0$.

iv) <u>The nontrivial solutions</u> $h^{-1}(0)\setminus(\{0\}\times[0,1])$ <u>in</u> $B_r \times [0,1]$ <u>contain</u> 2N <u>continua</u> $C^i_\pm$, $i=1,\ldots,N$, <u>where all</u> $C^i_\pm$ <u>are disjoint and</u> $\mathrm{cl}C^i_\pm \cap (\{0\} \times [0,1]) = (0,t_i)$ <u>and</u> $C^i_\pm \cap B_r \times \{0\} = x^i_\pm$ <u>and</u> $Ax^i_\pm = f(x^i_\pm)$, $i=1,\ldots,N$, $(x^i_+ = -x^i_-)$.

<u>REMARK 2.5.</u> E. Allgower [Al] has shown that the finite difference analogue (2.2) of (2.1) may have solutions which are not approximate solutions of (2.1). While any "right" solution of (2.2) can be shown to have certain symetries it is typical for the "wrong" solutions of (2.2) that these properties are lacking. Certainly, a reasonable numerical study of (2.1) should cope with the difficulty to select the "right" solutions. In view of this it seems to be noteworthy that one can show that those solutions of (2.2) which can be found by persueing the continua $C^i_\pm$ according to (2.4) are "right" solutions. As an example we have studied (2.2) with $f(x)=x^5$ for N=3. We have implemented the perturbation determined by G = -Id by virtue of (1.8) and (2.4) (See also (2.25) and (2.26)) and our numerical results provide a full justification of figure 1(a_2) and 2. Moreover , we have implemented the perturbation determined by $G = c \in R^N\setminus\{0\}$ (cf. (2.18) and (2.24)) and were led both to "right" and "wrong" solutions and especially found continua of "wrong" solutions as it is visulized in figure 2.
Thus, the problem of finding the "right" solution may be solved by chossing the "right" perturbation (here: G = -Id and not G = c).

<u>Proof of 2.4.:</u> Property (ii) is due to the definition of h. Hence, we can restrict to the case $t\in[0,1)$ and there $h(x,t)=0$ is equivalent to

$$Ax-kx-f(x) = 0, \quad k = t(1-t)^{-1}, \quad 0 \le k < \infty \tag{2.6}$$

which again is equivalent to $(k \neq 0)$

$$(Id-\mu A)x = -\mu f(x), \ \mu = k^{-1}, \ 0 < \mu < \infty . \tag{2.7}$$

It is wellknown that A has N distinct non-zero char. values (cf. [I-K], say $\mu_1,\ldots,\mu_N$, and, thus, each is of multiplicity one. Moreover, $-f(x) = o(|x|)$, and, therefore (2.7) has bifurcations at $\mu_1,\ldots,\mu_N$ according to [Kr]. Furthermore, Rabinowitz's global bifurcation theorem [R] (cf. (1.8)) together with $f(-x) = -f(x)$ yields continua $C^i_\pm$ ($C^i_+ = - C^i_-$) bifurcating from $k_i = \mu_i^{-1}$ for equation (2.6) and these satisfy one of the following:

- $C^i_\pm$ is unbounded in $R^n \times R$, or
- $\mathrm{cl}C^i_\pm \ni (0,k_j)$ $i \neq j$.

Now, let (x,k) be any nontrivial solution of (2.6), i.e. $Ax = f(x) + kx$. Then

$$|Ax| = |x^{2n+1} + kx| \leq |A||x|.$$

However, the norm $|\ |$ chosen has the property

$$|x^{2n+1} + kx| = |x||x^{2n} + ke| \geq |x|^2, \quad k \geq 0$$

whenever $|x| \geq 1$, where $e = (1,\ldots,1)^T$. Thus, we have

$$|x| \leq |A| \text{ for any solution } (x,k) \in R^n \times [0,\infty). \tag{2.8}$$

This proves (i)-(iii). Moreover, (cf. [R]) each branch $C^i_\pm$ inherits a "nodal structure" from the eigenspaces of A which implies that the $C^i_\pm$ are all disjoint and therefore each branch $C^i_\pm$ intersects $B_r \times \{0\}$ in a solution of $Ax=f(x)$.

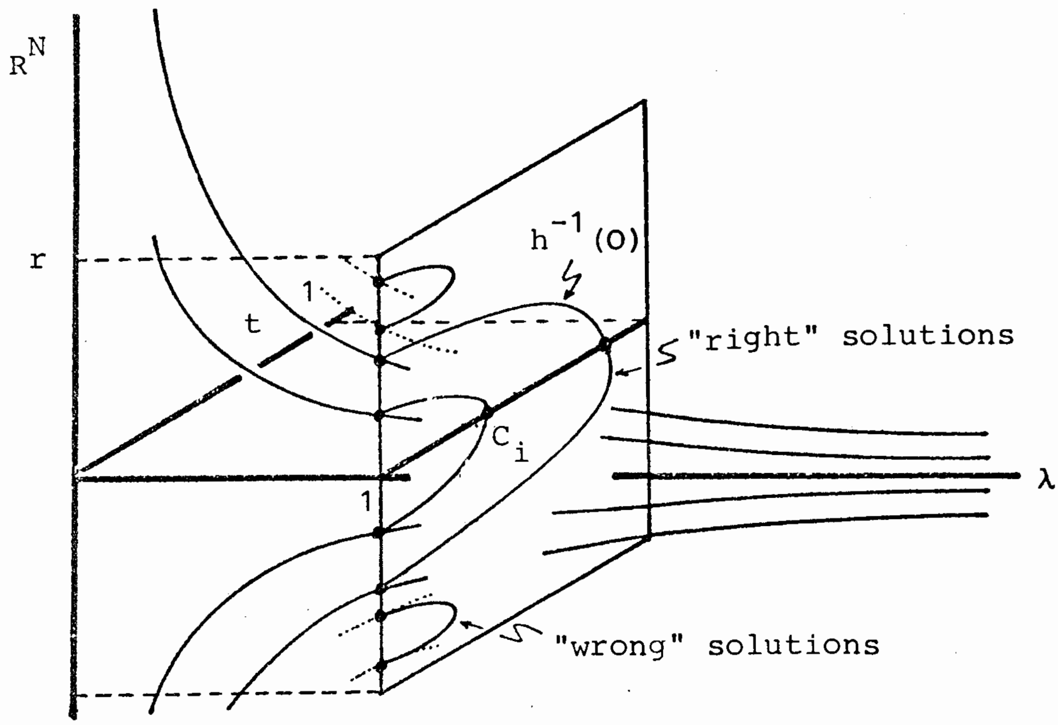


Figure 2 .

LEMMA 2.9. $(G = -A)$

Let $f(x) = x^{2n+1}$ and let $|x| = \max_{1\le i\le N} |x_i|$ where N is odd (N is the number of internal meshpoints for the uniform finite difference approximation of (2.1)).

Define

$$h(x,t) := (1-t)(Ax-f(x)) - tAx \quad \text{for} \quad 0 \le t \le 1 .$$

Then one has the following properties for $h^{-1}(0)$:

(i) There exists $r > 0$ such that $h^{-1}(0) \subset B_r \times [0,1]$;

(ii) $h^{-1}(0) \cap B_r \times [1/2,1] = \{0\} \times [1/2,1]$;

(iii) $h^{-1}(0)$ contains precisely one bifurcation point $(0,1/2)$ of multiplicity N and the continuum emanating from $(0,1/2)$ intersects $B_r \times \{0\}$ in solutions of $Ax = f(x)$.

Proof: One has that $h^{-1}(0) \cap B_r \times \{1\} = \{(0,1)\}$ due to the definition of h. Then equation $h(x,t) = 0$ is equivalent to $(t \neq 1)$:

$$(Id-kId)x = A^{-1}f(x), \; 0 \le k < \infty , \; k=t(1-t)^{-1} . \qquad (2.10)$$

However, (2.10) has only trivial solutions for $k \ge 1$: assume that x is any solution of (2.10) with $k > 1$ and assume without loss of generality that $x_1 \ge 0$; then the special structure of A would immediately imply that $x_2 \ge 0,\dots,x_N \ge 0$ and the conclusion follows from the wellknown fact that A^{-1} is a positive matrix. The characteristic value $k = 1$ is the only characteristic value of the bifurcation equation (2.10). Hence, it follows from [R] (N is odd) that (2.10) has a global branch C of nontrivial solutions emanating from $(0,1)$ (in the k-scale) which cannot intersect the trivial solutions in a point other than $(0,1)$. Moreover, we have the elementary estimates: $(0 \le k < 1)$ $|(1-k)Ax| = |f(x)| \le (1-k)|A||x|$, and, thus, $(1-k)|A| \ge |f(x)||x|^{-1} \ge |x|$, provided $|x| \ge 1$. Therefore, $r = \max\{|A|,1\}$ will satisfy (i) and (ii). Since we know already that C is unbounded in $R^N \times R$ we have (iii) from (i) and (ii).

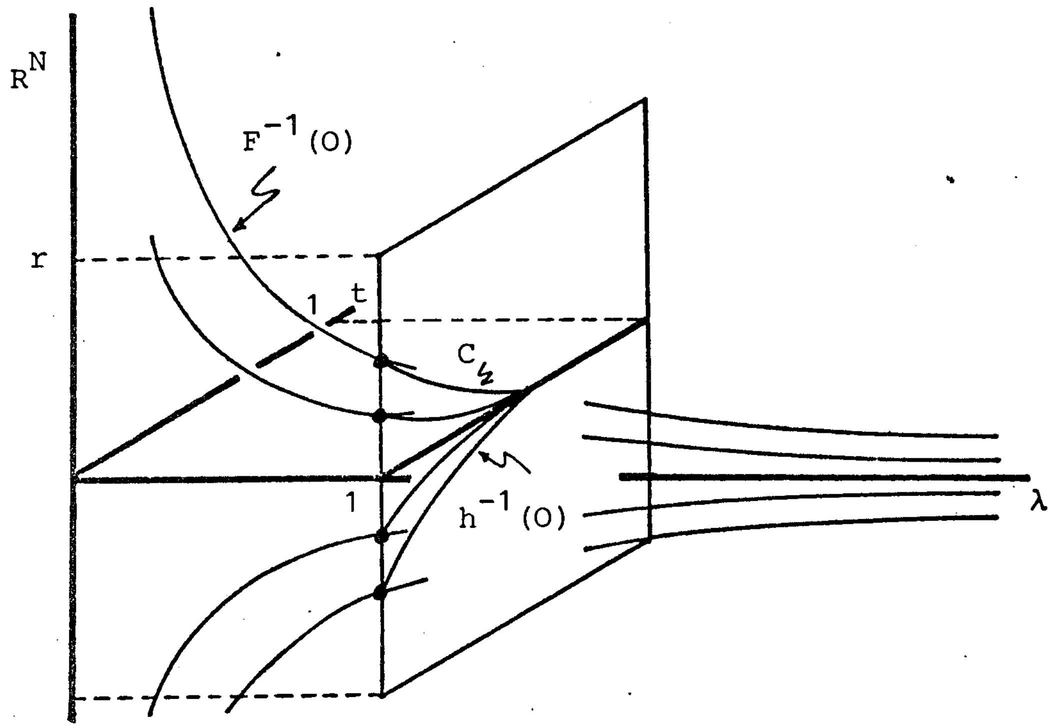


Figure 3.

REMARK 2.11. If one considers

(2.12) $(Id-kE)x = A^{-1}f(x)$ instead of (2.10), where $E = Id + diag(\varepsilon, 2\varepsilon, \ldots, N\varepsilon)$, then (2.12) has N distinct bifurcation points and one can obtain a similar analysis as in (2.4). This justifies the picture given in figure 3, where C is shown as a union of $2N$ symmetric nontrivial continua bifurcating from $(0,1/2)$ (in the t-scale).

REMARK 2.13. The previous lemmata can easily be generalized to more general nonlinearities f and also to problems of the form

$$\begin{cases} -\Delta u = \lambda f(u) \\ u|_{\partial\Omega} = 0 \quad , \quad \Omega = [0,1]^2 \end{cases} \tag{2.14}$$

and even more general elliptic boundary value problems exploiting the knowledge about a suitable finite dimensional approximation. For example, if Ω is the unit square in R^2 and if one uses a central difference approximation for (2.14) on a uniform grid with N internal meshpoints in Ω one obtains an approximate problem $Ax = \lambda f(x)$ whose linear part A is completely known with respect to eigenvalues, e.t.c. [I-K]. In this spirit we discuss a last example given in the next lemma.

Let $f: R_+ \to R$ be a C^1 function and satisfy the following hypothesis:

$$\begin{cases} f(s) = m_\infty s + \phi(s),\ m_\infty > 0,\ |\phi(s)| \le c \\ \qquad \text{for some } c > 0 \text{ and all } 0 \le s < \infty; \\ f(0) = 0,\ f'_+(0) > 0. \end{cases} \tag{2.15}$$

Especially let f_1, f_2 be two functions satisfying (2.15) and such that $f_1 \equiv f_2$ in a neighborhood of zero and infinity and, moreover, let $f_1(s^0) < 0$ for some $s^0 > 0$ and $f_2(s) \ge 0$ for all $0 \le s < \infty$ (see figure 4).

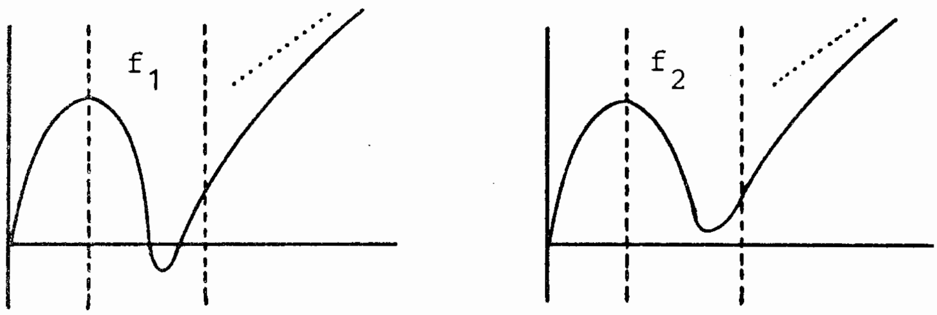


Figure 4.

As usual we denote by $\bar{f}: R_+^N \to R^N$ the mapping $\bar{f}(x_1,\ldots,x_N) := (f(x_1),\ldots,f(x_N))$ and write again f.

LEMMA 2.16. $(G = Ax - \lambda g(x))$

Let $f_{1,2}: R_+^N \to R^N$ be as above. Let $Ax = \lambda f_{1,2}(x)$ be the finite difference approximation of (2.1) or (2.14) on a uniform grid of N internal meshpoints. Then one has:

(i) $Ax = \lambda f_{1,2}(x)$ has bifurcation from 0 at $\lambda_0 := f'_+(0)^{-1}\cdot\lambda_1$ and bifurcation from ∞ at $\lambda_\infty := m_\infty^{-1} \cdot \lambda_1$ for positive solutions $(x_i \ge 0,\ i=1,\ldots,N)$, where λ_1 is the smallest characteristic value of A.

(ii) Let $C_o(f_{1,2})$ respectively $C_\infty(f_{1,2})$ denote the continua bifurcating from 0 respectively from ∞. Then

$$C_o(f_1) \cap C_\infty(f_1) = \emptyset$$

$$C_o(f_2) = C_\infty(f)$$

and there exists a $\lambda_* > 0$ such that $Ax = \lambda f_2(x)$ has no positive solution with $\lambda \geq \lambda_*$.

(iii) Let $h(x,t) = (1-t)(Ax-\lambda_* f_1(x)) + t(Ax-\lambda_* f_2(x))$. Then $h^{-1}(0)$ connects $C_o(f_1)$ with $C_\infty(f_1)$ (see figure 5).

Proof: Observe that $h(x,t) = Ax - g_t(x)$, where g_t satisfies (2.15) for all $t \in [0,1]$ and where $g_t(x) = (1-t) \cdot f_1(x) + t \cdot f_2(x)$. Properties (i) and (ii) are proved in [A-H] for elliptic boundary value problems and their ideas can be transferred to the approximation $Ax = \lambda f_{1,2}(x)$. For (iii), observe that the bifurcation analysis of Ambrosetti-Hess [A-H] applies to g_t for each t and one can show their 'a priori estimates' to conclude (iii). $Ax = g_t(x)$ can be looked at as the finite difference approximation of

$$\begin{cases} -\Delta u = g_t(u) \\ u_{|\partial\Omega} = 0 . \end{cases} \tag{2.17}$$

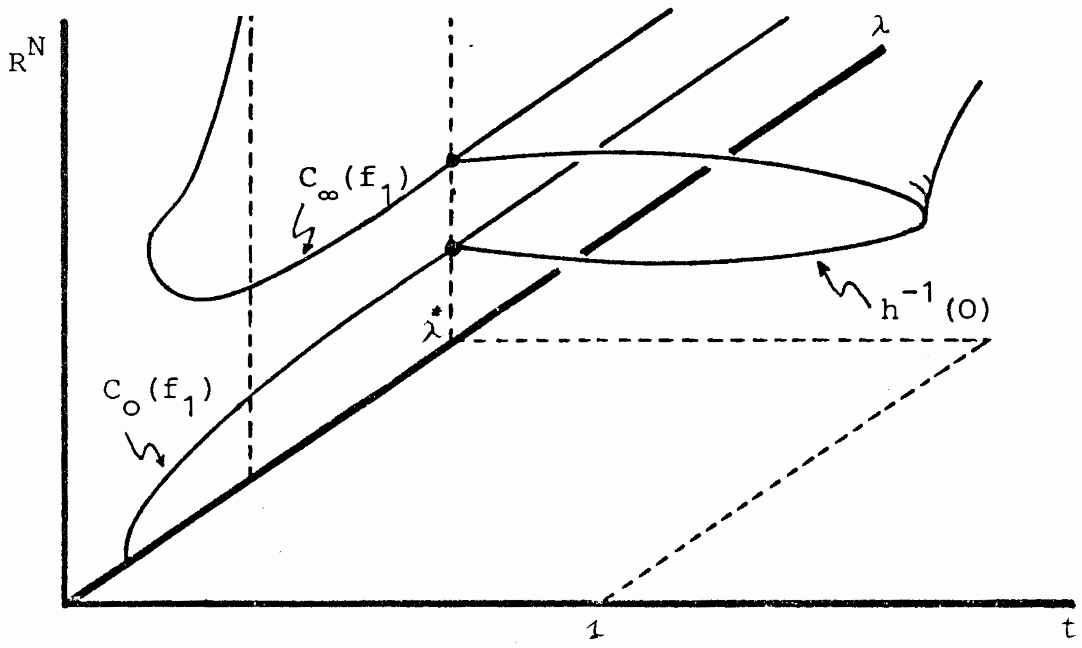


Figure 5.

Similar to the remark (2.5) one should observe that $Ax = \lambda f_1(x)$ may allow "wrong" solutions, i.e. solutions which are not approximate solutions of (2.17). However, as is obvious from figure 5, one can create a continuum of positive solutions which bifurcates at λ_1 and then connects with $C_\infty(f_1)$. Thus, in presence of "wrong" solutions, $h^{-1}(0)$ provides a selection of "right" solutions. The general idea of the examples above is to design a perturbation in such a way that the resulting equations again can be understood as an approximation to an appropriate differential equation:

$$\left.\begin{array}{lll}(2.6) & \hat{=} & -u'' = ku + f(u), \quad 0 \le k < \infty \\ (2.10) & \hat{=} & -(1-k)u'' = f(u), \quad 0 \le k < \infty \\ (2.16) & \hat{=} & -u'' = g_t(u), \quad 0 \le t \le 1 \end{array}\right\} \quad u(0) = u(1) = 0.$$

For the choices $G \equiv c \in R^N \setminus \{0\}$ and $G \equiv P \circ F,\ P \in GL\_$, we restrict to an instructive example where $N = 2$.

(2.18) Let $f(x) = x^3$ and choose $c = \binom{1}{0}$ and define

$$h(x,t) = (1-t)(Ax-x^3) + tc,$$

where

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Then $h(x,t) = 0$ is equivalent to $(\lambda = t(1-t)^{-1})$

$$(2.19) \qquad \begin{array}{lll} 2x_1 - x_2 - x_1^3 + \lambda = 0 & : & I_\lambda \\ -x_1 + 2x_2 - x_2^3 \qquad = 0 & : & II \end{array}.$$

Figure 6 pictures the curves $I_\lambda = 0$ and $II = 0$.

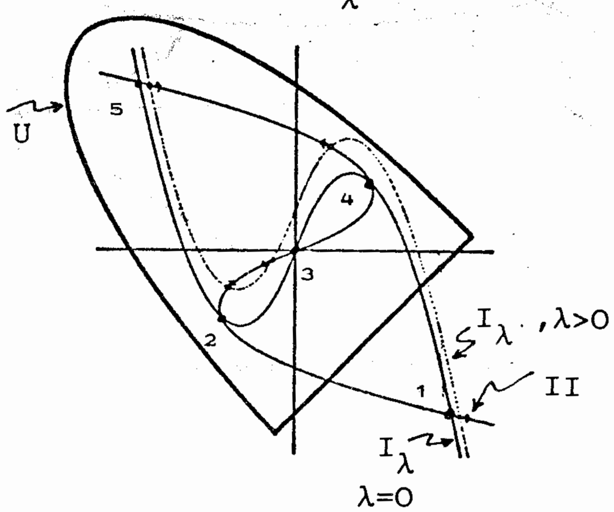


Figure 6.

The five intersection points represent zeroes of $Ax-x^3 = 0$. Obviously, if one chooses U as indicated one would have that $h^{-1}(0) \cap U \times [0,1] = \emptyset$, and, hence, the figure 7a is justified:

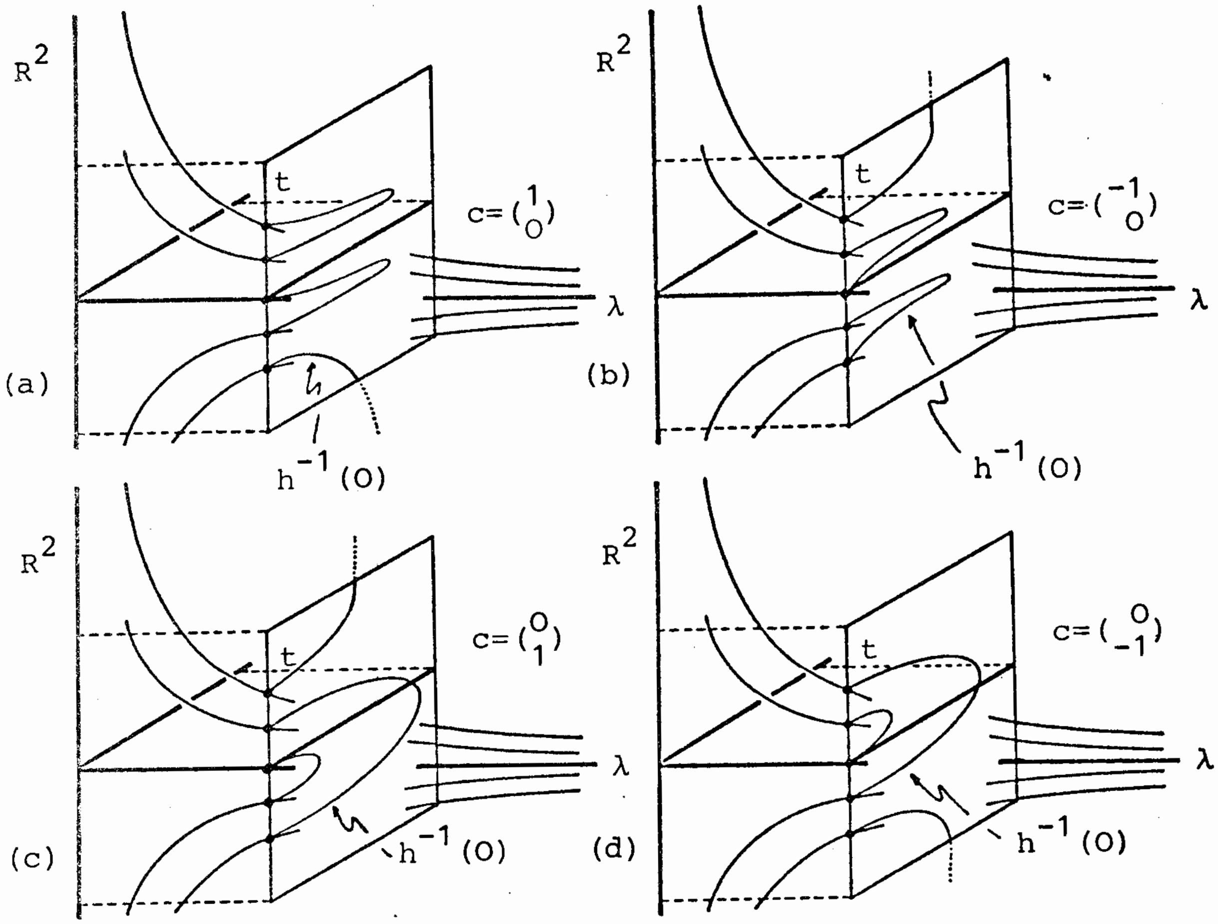


Figure 7.

We observe that for each choice of c two pairs of nontrivial continua of solutions of $Ax = \lambda x^3$ obtain an artificial connection via $h^{-1}(0)$. However, in each case $h^{-1}(0)$ contains a component which goes to ∞ without providing an artificial connection. This phenomenon demonstrates a typical difficulty: For any $r > 0$ large enough one has that $h^{-1}(0) \cap \partial B_r \times [0,1] \neq \emptyset$. However, choosing a neighborhood U as indicated in figure 6 one has 'a priori estimates'. A similar analysis can be made for the case where $G = P \circ F$ and where P is typically given by

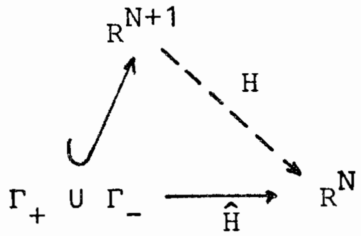
$$P = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} .$$

SETTING OF TOPOLOGICAL PERTURBATIONS

Let $F: R^{N+1} \to R^N$ be continuous and let $\Gamma_+, \Gamma_- \subset R^{N+1}$ be disjoint sets which are the closure of open sets in R^{N+1} and let $G: \Gamma_- \to R^N$ be given. Define

$$\hat{H}(x,\lambda) =: \begin{cases} F(x,\lambda) & \text{on } \Gamma_+ \\ G(x,\lambda) & \text{on } \Gamma_- \end{cases} \tag{2.20}$$

and let $[\hat{H}]$ denote the class of extensions of $\hat{H}$ to the whole of R^{N+1}. Observe that R^N is an AR (absolute retract) and, therefore, (cf. [Du]) there exists an H such that the following diagram is commutative



If $\Gamma_+ = R^N \times (-\infty, \lambda_1]$ and $\Gamma_- = R^N \times [\lambda_2, \infty)$ then a typical choice for H in $R^N \times (\lambda_1, \lambda_2)$ might be

$$H(x,\lambda) = (\lambda-\lambda_2)(\lambda_1-\lambda_2)^{-1}F(x,\lambda_1) + (\lambda-\lambda_1)(\lambda_2-\lambda_1)^{-1}G(x,\lambda_2) \tag{2.21}$$

which has been used in (2.4), (2.9), (2.16) and (2.18). If Γ_+, Γ_- are more general (e.g. cubes) then the setting of homotopy is not any more appropriate and one has to use the setting of extensions. Typically, one will choose Γ_+, Γ_- then in such a way that for any extension H certain PL-approximations are uniquely determined by F and G (see (2.23)).

Examples (2.4), (2.9), (2.16) and (2.18) demonstrate how topological perturbations H of a given problem $F(x,\lambda) = 0$ may create artificial connections for solutions in $F^{-1}(0)$. Since we are mainly interested in the computational aspect in view of continuation and simplicial path following algorithms we restrict in the following to a generic situation, i.e. in the C^∞-category we may assume that 0 is a regular value of H and in the PL-category we choose $\bar{\varepsilon} = (\varepsilon, \varepsilon^2, \ldots, \varepsilon^n)$ along with a triangulation T such that $\bar{\varepsilon}$ is a regular value in the PL sense. Then in both cases $H^{-1}(0)$ (respectively $H_T^{-1}(\bar{\varepsilon})$) is a collection of 1-dimensional smooth (respectively PL) manifolds. For reasons of length we consider in the following only the PL case. The C^∞ case, however, has a correspondance word by word.

Let T be any triangulation of R^{N+1}. Then T and R^{N+1} have a decomposition

$$\begin{aligned} T^+ &= \{\sigma\in T_{n+1}: \sigma\subset\Gamma_+\}, & \Gamma_+^T &= \bigcup_{\sigma\in T^+}\sigma \\ T^- &= \{\sigma\in T_{n+1}: \sigma\subset\Gamma_-\}, & \Gamma_-^T &= \bigcup_{\sigma\in T^-}\sigma \\ T^a &= \{\sigma\in T_{n+1}: \sigma\not\subset T^+\cup T^-\}, & \Gamma_a^T &= \bigcup_{\sigma\in T^a}\sigma . \end{aligned} \tag{2.22}$$

Here Γ_a^T is the subset of R^{N+1} where the perturbation H will create artificial connections of solutions of $F^{-1}(0)$. We have the following elementary lemma:

<u>LEMMA 2.23.</u> (Properties of H_T)

<u>Let</u> Γ_+, Γ_-, F, G <u>be as above and let</u> T <u>be a triangulation of</u> R^{N+1}

(i) <u>Assume that</u>

vertices(T^+) ∪ vertices(T^-) = vertices (T).

<u>Then the natural PL-map</u> H_T <u>which is defined for any extension</u> $H\in[\hat{H}]$ of $\hat{H}: \Gamma_+\cup\Gamma_-\to R^N$ <u>is unique and is completely determined by</u> $F_{|T^+}$ <u>and</u> $G_{|T^-}$.

(ii) <u>Let</u> $\bar{\varepsilon} = (\varepsilon,\varepsilon^2,\ldots,\varepsilon^n)$ <u>and</u> ε <u>small then</u> $H_T^{-1}(\bar{\varepsilon}) = M_H$ <u>is a collection of PL-one-manifolds with</u>

$M_H \cap \Gamma_+^T$ <u>approximates</u> $F^{-1}(0) \cap \Gamma_+^T$

$M_H \cap \Gamma_-^T$ <u>approximates</u> $G^{-1}(0) \cap \Gamma_-^T$.

<u>Let</u> $m_H \in M_H$ <u>be a component intersecting</u> Γ_a^T <u>then</u>

$$m_H \cap \Gamma_a^T \cong \begin{cases} [0,1] & \text{or} \\ [0,1),(0,1) & \text{or} \\ S^1 \end{cases} \qquad (\cong \text{ isomorphic})$$

(iii) Let $U \subset R^N$ be a bounded and open neighborhood and assume that $H^{-1}(0) \cap \partial U \times [\lambda_1,\lambda_2] = \emptyset$. Let $\Gamma_+ = R^N \times (-\infty,\lambda_1]$ and $\Gamma_- = R^N \times [\lambda_2,\infty)$ and let T be a triangulation of $R^N \times R$ sufficiently fine and parallel to R^N and compatible with $R^N \times [\lambda_1,\lambda_2]$. Then

$$m_H \cap \Gamma_a^T \cap (U \times [\lambda_1,\lambda_2]) \cong \begin{cases} [0,1] \text{ , or} \\ S^1 \end{cases}$$

and the $[0,1]$-components connect solutions in $(F_T^{-1}(\bar{\varepsilon}) \cap \Gamma_+^T) \cup (G_T^{-1}(\bar{\varepsilon}) \cap \Gamma_-^T)$ and, moreover,

$$\deg(F(\cdot,\lambda_1),U,0) = \deg(G(\cdot,\lambda_2),U,0).$$

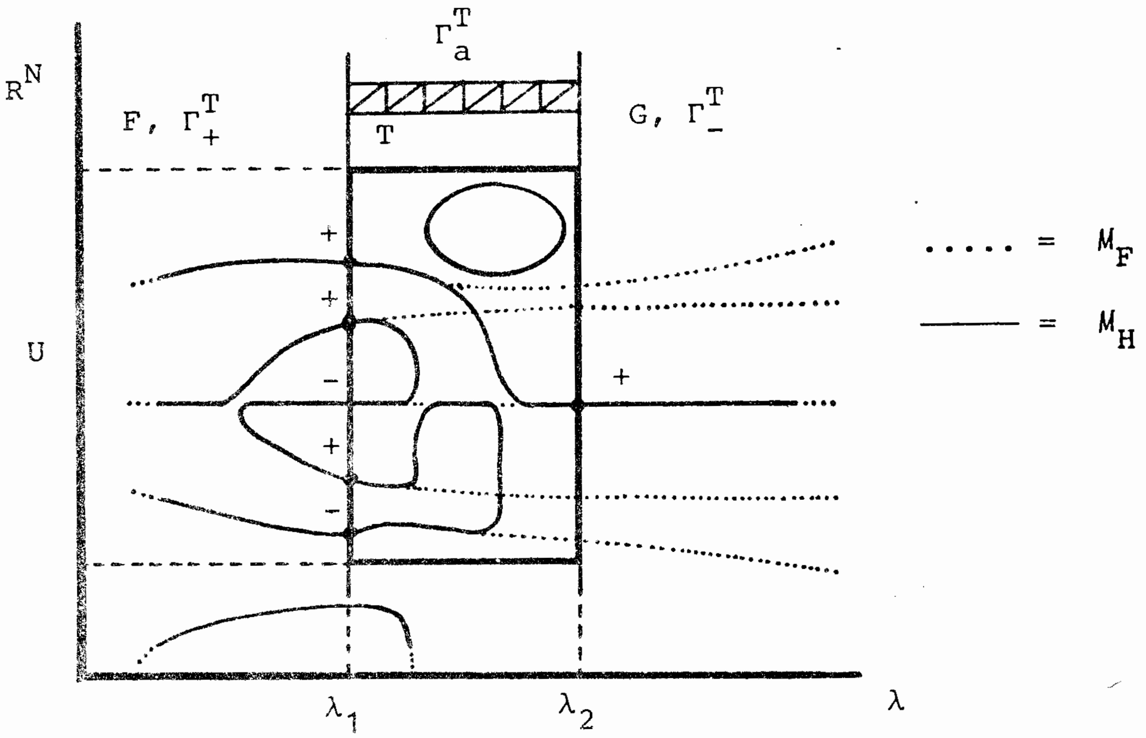


Figure 8.

(T is compatible with $R^N \times [\lambda_1,\lambda_2]$ if $\Gamma_a^T = R^N \times [\lambda_1,\lambda_2]$)

Typically, a proof for the existence of a connection via H will be to verify the assumptions of (2.23. iii), i.e. find U bounded and open such that $H^{-1}(0) \cap \partial U \times [\lambda_1,\lambda_2] = \emptyset$. However, this approach restricts the freedom of choice for $G: \Gamma_- \to R^N$ in a very typical manner:

(2.24) EVEN NUMBER OF SOLUTIONS (Γ_+, Γ_- half spaces)

If one chooses $G \equiv c \in R^N \setminus \{0\}$ or $G = P \circ F$, $P \in GL_-$, and finds U open and bounded such that $H^{-1}(0) \cap U \times [\lambda_1, \lambda_2] = \emptyset$ then

$$\deg(F(\cdot,\lambda_1),U,0) = 0, \quad \text{i.e.}$$

$F_T^{-1}(\bar{\varepsilon}) \cap R^N \times \{\lambda_1\}$ is an even number (see figure 1(b_1) for $G \equiv c$ and figure 1(b_2) for $G = P \circ F$).

If $F(\cdot,\lambda_1)^{-1}(0)$ has generically an odd number of solutions in some domain of interest then an application of the choices $G \equiv c$ or $G = P \circ F$ may be troublesome in theory, because in order to find an appropriate U (cf. figure 6) one would need 'a priori' knowledge about the solutions. In this case one would rather use a more sophisticated perturbation:

(2.25) ODD NUMBER OF SOLUTIONS (Γ_+, Γ_- halfspaces)

If $F(\cdot,\lambda_1)^{-1}(0)$ has generically an odd number of solutions in U then $\deg(F(\cdot,\lambda_1),U,0) \neq 0$.

If one chooses $G: \Gamma_- \to R^N$ such that $G^{-1}(0) \subsetneqq F^{-1}(0)$ and $H^{-1}(0) \cap \partial U \times [\lambda_1, \lambda_2] = \emptyset$ then generically one has

$$\text{number of sol. in } F_T^{-1}(\bar{\varepsilon}) \cap U \times \{\lambda_1\} -$$
$$\text{number of sol. in } G_T^{-1}(\bar{\varepsilon}) \cap U \times \{\lambda_2\}$$
$$= 2M \quad \text{for some } M \in N$$

and $H_T^{-1}(\bar{\varepsilon}) \cap U \times [\lambda_1, \lambda_2]$ contains M components of type [0,1] each of which connects a pair of solutions in $F_T^{-1}(\bar{\varepsilon}) \cap U \times \{\lambda_1\}$.

(2.25) is another explanation of (2.4), (2.9) and (2.16) and is visualized in figure 1(a_2,a_3,b_1,c_6). Moreover, in view of this it becomes clear how (2.4), (2.9) and (2.16) could be exploited for path following algorithms. In both cases (2.24) and (2.25) are easy consequences of (1.7). To illustrate (2.25) we generalize (2.4), (2.9) and (2.16):

LEMMA 2.26. *Let* $F: R^{N+1} \to R^N$ *have the special form* $F(x,\lambda) = Ax - f(x,\lambda)$, *where* $A \in GL$ *and where* $f(x,\lambda)$ *is* $o(|x|)$ *uniformly on bounded* λ-*intervals, i.e.* $F(0,\lambda) = 0$ *for all* λ. *Let* $\Gamma_+ = R^N \times (-\infty,\lambda_1]$, $\Gamma_- = R^N \times [\lambda_2,\infty)$ $(\lambda_1 < \lambda_2)$ *and let* $G: \Gamma_- \to R^N$ *be one of the following:*

(2.27) $G(x,\lambda) = -x$ *and* sign det $A = -(-1)^N$.

(2.28) $G(x,\lambda) = -Ax$ *and* N *is odd.*

(2.29) $G(x,\lambda) = Ax - g(x,\lambda)$ *and* $G^{-1}(0) = \{0\} \times [\lambda_2,\infty)$.

Let $H(x,\lambda) := (\lambda-\lambda_2)(\lambda_1-\lambda_2)^{-1}F(x,\lambda_1) + (\lambda-\lambda_1)(\lambda_2-\lambda_1)^{-1}G(x,\lambda_2)$. *Assume that* $H: R^N \times [\lambda_1,\lambda_2] \to R^N$ *satisfies*

(2.30) $H^{-1}(0) \cap \partial U \times [\lambda_1,\lambda_2] = \emptyset$ *for some bounded open neighborhood* U in R^N.

Let T *be a triangulation of* $R^N \times [\lambda_1,\lambda_2]$ *sufficiently fine. Then we have with the choices* (2.27) *or* (2.28) *and if* $0 \in U$ *that* $H^{-1}(0)$ *bifurcates in* $[\lambda_1,\lambda_2]$ *from the trivial solutions and the global branches connect the trivial solution with* $F^{-1}(0) \cap R^N \times \{\lambda_1\}$ *and the number of solutions in the latter set is generically odd, e.g.* $2M + 1$ $(M \geq 1)$. *Furthermore* $H_T^{-1}(\bar{\varepsilon}) \cap U \times [\lambda_1,\lambda_2]$ *contains at least* M *components of type* $[0,1]$ *each of which connects a pair of solutions in* $F_T^{-1}(\bar{\varepsilon}) \cap U \times \{\lambda_1\}$.

If G *is chosen as in* (2.29) *and if* $0 \notin U$ *then* $F^{-1}(0) \cap U \times \{\lambda_1\}$ *has generically an even number of solutions, e.g.* $2M$, *and* $H_T^{-1}(\bar{\varepsilon}) \cap U \times [\lambda_1,\lambda_2]$ *contains* M *components of type* $[0,1]$ *each of which connects a pair of solutions in* $F_T^{-1}(\bar{\varepsilon}) \cap U \times \{\lambda_1\}$.

Proof: Observe that with the choices (2.27) and (2.28) one has that $\deg(H(\cdot,\lambda_1),B_\varepsilon,0) = -\deg(H(\cdot,\lambda_2),B_\varepsilon,0)$ (ε sufficiently small). Furthermore, completely labelled simplices in $U \times \{\lambda_2\}$ with respect to H are unique because $-Id$ and $-A$ are isomorphisms. Now the assertions of the first part follow easily from (1.7), (1.8), the global bifurcation theorem of Rabinowitz [R] and arguments similar to those in the proofs of (2.4) and (2.9). If G is chosen according to (2.29) then $\deg(F(\cdot,\lambda),U,0) = 0$ because $0 \notin U$, and the assertion follows again from (1.7).

Obviously, (2.26) admits various extensions and generalizations in the spirit of (2.24) and (2.25).

In [P-P] an extensive discussion of the perturbation $G = P \circ F$, $P \in GL_-$, is given. Special interest is devoted to bifurcation phenomena in the setting of

- finding all bifurcation branches or
- passing through a bifurcation point.

If one chooses Γ_+ (resp. Γ_-) to be a cube in R^{N+1} and Γ_- (resp. Γ_+) to be the complement of a cube similar results as in the previous discussion can be derived. However, there is one major difference and advantage: any component of $H_T^{-1}(\bar{\varepsilon}) \cap \Gamma_a^T$ is either S^1 or $[0,1]$. Rather than going into details we restrict to two typical choices which are indicated in figure 9 and which deal with the resolution of bifurcations:

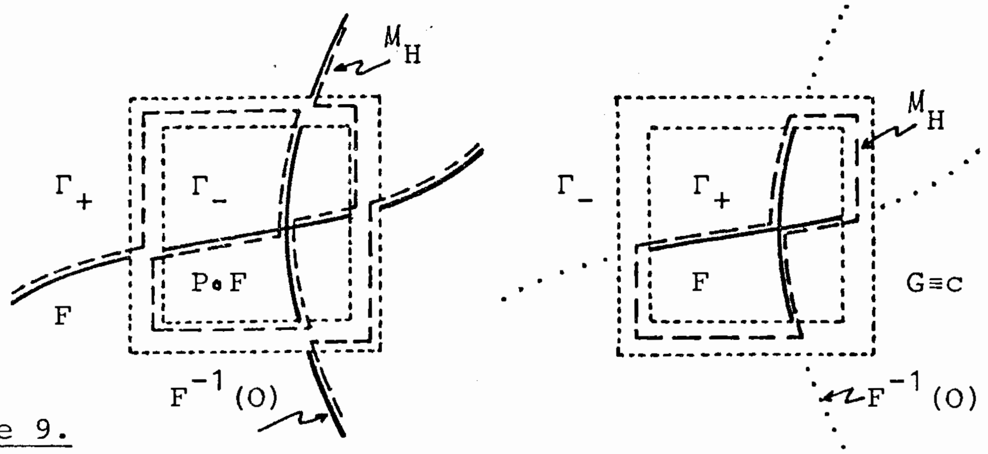


Figure 9.

Finally, we note a property which is common both to the perturbations $G \equiv c$ and $G = P \circ F$. Obviously, if $H(x,t) = (1-t)F(x,\lambda_1) + t\cdot c$, then $H(x,t) = 0$ iff $F(x,\lambda_1) = -\mu c$, $\mu = t(1-t)^{-1}$ and $0 \le t < 1$. Hence, $H^{-1}(0)$ is given by $F^{-1}(R_-\cdot c)$ and choosing c with an appropriate direction one can pilot the algorithm subject to this observation. For the case where $G = P \circ F$ we restrict for the sake of simplicity to the special choice where $P \circ F = (F_1,\dots,-F_i,\dots,F_N)$.

Then with $H(x,\lambda) = (\lambda-\lambda_2)(\lambda_1-\lambda_2)^{-1}F(x,\lambda) + (\lambda-\lambda_1)(\lambda_2-\lambda_1)^{-1}P\circ F(x,\lambda)$, $\lambda \in [\lambda_1,\lambda_2]$ we have that

$$H(x,\lambda) = (F_1(x,\lambda),\ldots,(2\lambda-\lambda_1-\lambda_2)(\lambda_1-\lambda_2)^{-1}F_i(x,\lambda),\ldots,F_N(x,\lambda))$$

and $H^{-1}(0) \supset F^{-1}(R\cdot e_i)$ since $2\lambda-\lambda_1-\lambda_2 = 0$ for $\lambda = (\lambda_1+\lambda_2)/2$, where e_i is the i-th unit vector.

3. TOPOLOGICAL PERTURBATIONS II

Our aim here is to introduce and discuss two ideas in connection with simplicial path following algorithms which are set up to accelerate pivoting schemes. In the line of our previous perturbation techniques the following can be understood as a perturbation given by the following data:

(3.1)
- $F: R^{N+1} \to R^N$
- $\Gamma_+=R^N \times (-\infty,\lambda_1]$, $\Gamma_- = R^N \times [\lambda_2,\infty)$
- $G: \Gamma_- \to R^N$
- The design of G will be such that $G^{-1}(0) \cong F^{-1}(0)$ and this will be typically achieved by the choice $G = F \circ \Phi$, where $\Phi: R^{N+1} \to R^{N+1}$ is an affine isomorphism.

Our exposition will be based on triangulations T of R^{N+1} which are of K-type and which are parallel to the euclidean base of R^{N+1}. If T is such a triangulation then we have that certain λ-levels $R^N \times \{\lambda_k\}$ inherit an N-dimensional triangulation T_{λ_k} of type K (see figure 10).

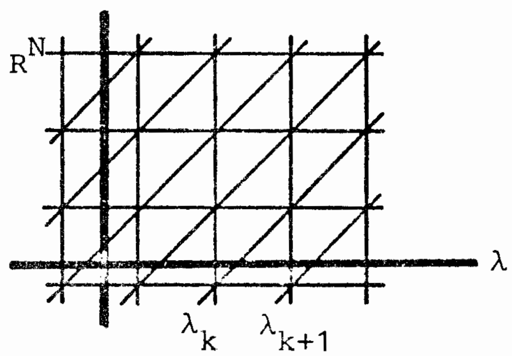


Figure 10.

(3.2) ACCELERATION BY PREDICTION

Let T be a triangulation as above and let $R^N \times \{\lambda_1\}$, $R^N \times \{\lambda_2\}$ be two consecutive λ-levels in T. Let $F: R^{N+1} \to R^N$ be given and assume that $\sigma_1 \in T_{\lambda_1}$ is an N-dimensional completely labelled simplex carrying the zero (x_1, λ_1) of F_T. Assume further that (x_2^*, λ_2) is an estimate of a zero for F_T in $R^N \times \{\lambda_2\}$, e.g. (x_2^*, λ_2) may be obtained by extrapolation (linear or nonlinear) from the known (i.e. already computed) solutions of $F_T^{-1}(0)$.

We define an affine isomorphism in the spirit of (3.1):

$$(3.3) \quad \begin{cases} \Phi: R^N \times R \to R^N \times R \\ \Phi(x,\lambda) := (x + (x_2^* - x_1), \lambda) \\ \text{and let } \Gamma_+, \Gamma_- \text{ be the halfspaces defined} \\ \text{in (3.1) and set} \\ G := F \circ \Phi \end{cases}$$

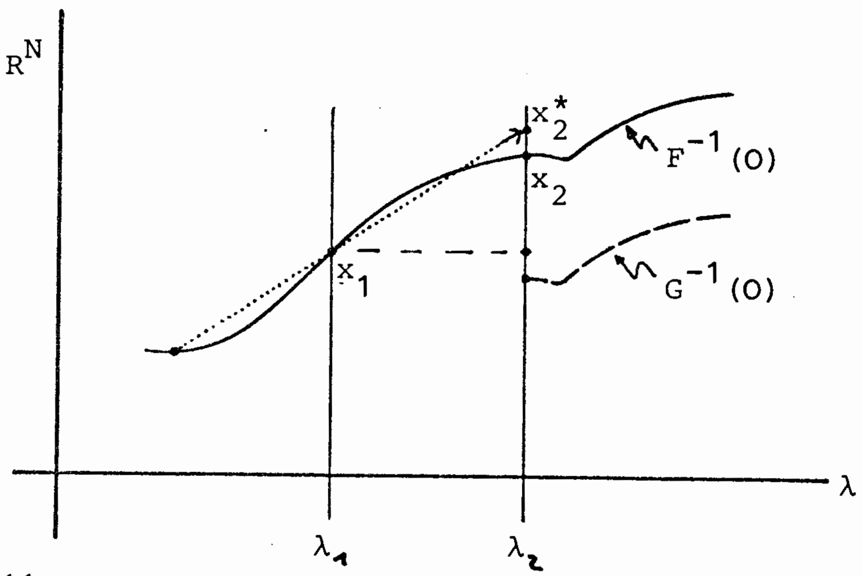


Figure 11.

Now (x_1, λ_2) is an estimate for a zero of G. Observe that (x_1, λ_2) is a point in T which is door-to-door with (x_1, λ_1). Therefore the number of pivoting steps the algorithm needs to get from $R^N \times \{\lambda_1\}$ to $R^N \times \{\lambda_2\}$ can be expected to be much smaller than in the unperturbed situation. Now let $(\bar{x}_2, \lambda_2)$

be the zero of G_T obtained by following the pivoting scheme for H with initial simplex σ_1: $(\bar{x}_2,\lambda_2)\in\bar{\sigma}_s\in ch_H^T(\sigma_1)$. We have the following lemmata using notation as above and assuming that $\bar{\sigma}_s \in ch_H^T(\sigma_1)$ exists in $R^N \times \{\lambda_2\}$:

LEMMA 3.4.

Let $\hat{T} = \Phi(T)$ be the triangulation induced by T and Φ and let $\sigma_s = \Phi(\bar{\sigma}_s) \in \hat{T}_n$, where $\bar{\sigma}_s \in T_n$ and $\bar{\sigma}_s \in ch_H^T(\sigma_1)$. Then

$$(x_2,\lambda_2) := \Phi(\bar{x}_2,\lambda_2) = (\bar{x}_2 + (x_2^*-x_1),\lambda_2)$$

is a zero of $F_{\hat{T}}$ and σ_s is a completely labelled simplex for F in $R^N \times \{\lambda_2\}$ with respect to $\hat{T}$.

We note the important equivalence:

LEMMA 3.5.

Let $\Psi: R^N \times R \to R^N \times R$ be the map

$$\Psi(x,\lambda) := (x + (\lambda-\lambda_1)(\lambda_2-\lambda_1)^{-1}(x_2^*-x_1),\lambda) .$$

Define $\tilde{T}$ a triangulation of $R^N \times [\lambda_1,\lambda_2]$ by $\tilde{T} = \Psi(T)$, where T is the original triangulation. Then $\tilde{T}_{\lambda_1} = T_{\lambda_1}$ and $\tilde{T}_{\lambda_2} = \hat{T}_{\lambda_2}$ and

$$\sigma_s \in ch_F^{\tilde{T}}(\sigma_1).$$

Thus, whether the perturbation is performed in F and T is fixed or is performed in T and F is fixed is equivalent. It seems likely that an implementation should make use of (3.5). Actually, our implementation has been the one of (3.5) and will be discussed in a difficult numerical problem in section 4.

Obviously, one can iterate this acceleration technique and this is visualized in figure 12.

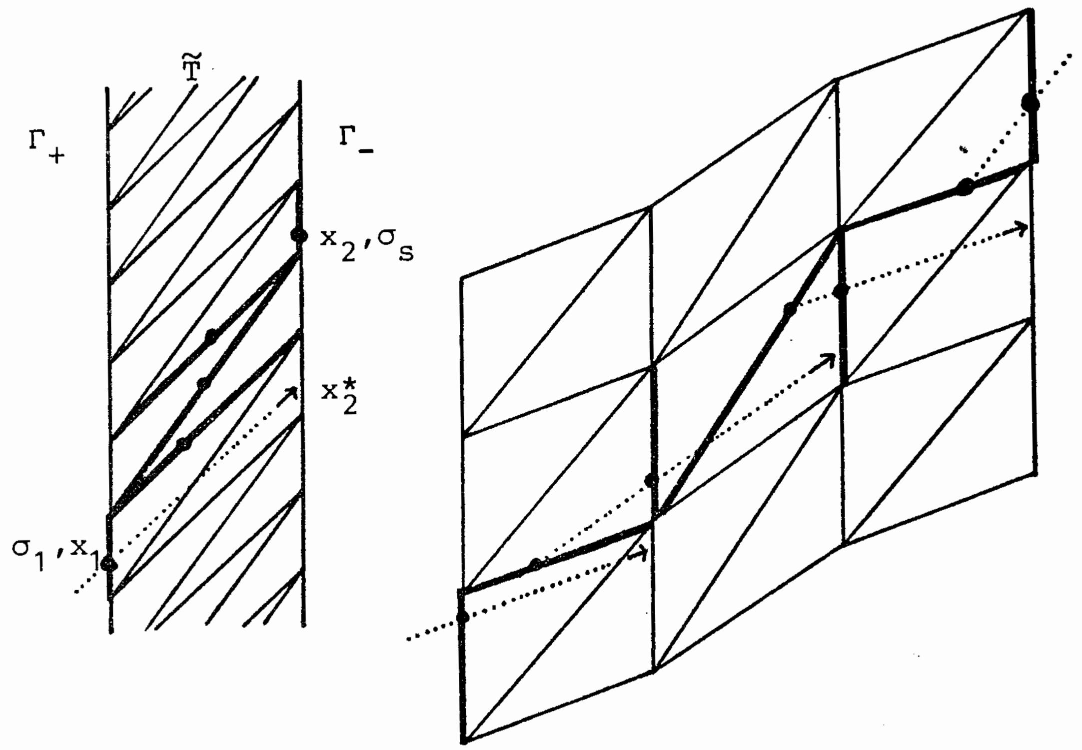


Figure 12.

We note, however, that the technique will perform safely only as long as $ch_F^T(\sigma_1)$ intersects $R^N \times \{\lambda_2\}$ in a completely labelled simplex σ_s and $\{\sigma_1, \ldots, \sigma_s\} \cap R^N \times (-\infty, \lambda_1) = \emptyset$, i.e. $F^{-1}(0)$ has no turning point in $[\lambda_1, \lambda_2]$. In presence of such points the acceleration technique may lead to cycling. This undesirable phenomenon could be avoided by taking into account all transformations used before level λ_1. A strategy much easier to implement would be, however, to stop the acceleration procedure in presence of a turning point close to that point and process the algorithm in the usual K-type triangulation.

Using finite difference approximations on a uniform grid of 25 internal mesh points the following example has been computed (see figure 13) both with and without the acceleration technique by the simplicial algorithm:

$$\begin{cases} -\Delta u = \lambda \exp(u) , & \lambda \in R_+ \\ u_{|\partial\Omega} = 0 & \Omega = [0,\pi]^2 \end{cases} .$$

The mesh size of the triangulation was 10^{-2}. In figure 13 we have subdivided the continuum S in three segments $S_1 \cup S_2 \cup S_3$. For each segment the computing time without acceleration has been normalized to 1. Then the computing time with acceleration was:

$$S_1 \cup S_2 \cup S_3 \hat{=} 0.2 \;, \quad S_1 \hat{=} 0.2 \;, \; S_2 \hat{=} 0.7 \;, \quad S_3 \hat{=} 0.1$$

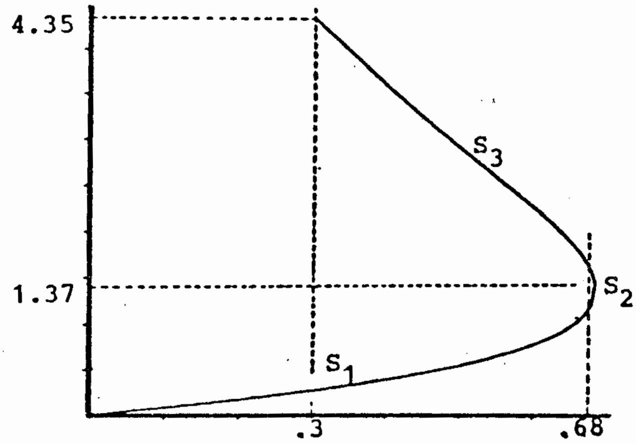


Figure 13.

3.6 VARIABLE MESH SIZE

Here we describe how one can use the setting of topological perturbations to create triangulations with a variable mesh size. More precisely, let T and T' be affinely isomorphic triangulations of R^{N+1} and let $\sigma_1 \in T_n$ be completely labelled. The aim is to find $\sigma_1' \in T_n'$ which is completely labelled and which approximates the same zero of $F: R^{N+1} \to R^N$ as σ_1 does.

Let T be a K-type triangulation of R^{N+1}, $R^N \times \{\lambda_1\}$ and $R^N \times \{\lambda_2\}$ two consecutive λ-levels in T and let $\sigma_1 \in T_{\lambda_1}$ be a completely labelled simplex carrying the zero (x_1,λ_1) of F_T. We define an affine isomorphism:

(3.7)
$$\left\{\begin{array}{l} \Phi: R^N \times R \to R^N \times R \\ \Phi(x,\lambda) := (r\cdot(x-x_1) + x_1, \lambda-(\lambda_2-\lambda_1)) \\ \text{for any } 0 < r \in R \\ \text{and let } \Gamma_+,\Gamma_- \text{ be the halfspaces defined} \\ \text{in (3.1) and let} \\ G = F \circ \Phi \;, \text{ i.e. } G^{-1}(0) \cap R^N \times [\lambda_2,\infty) \;\cong \\ \qquad\qquad F^{-1}(0) \cap R^N \times [\lambda_1,\infty) \end{array}\right.$$

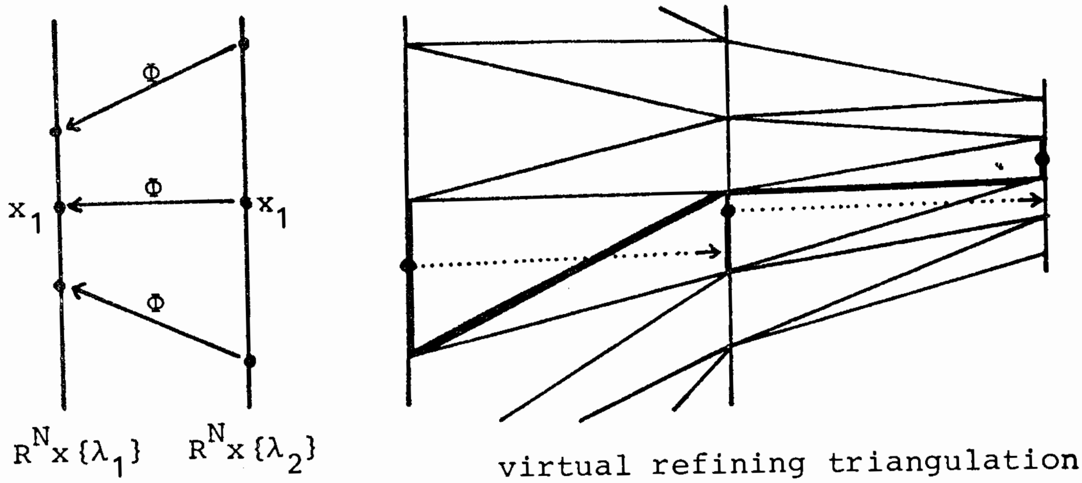


Figure 14.

Assume now that the pivoting scheme finds a completely labelled simplex $\sigma_s \in ch_H^T(\sigma_1)$ in $R^N \times \{\lambda_2\}$ which carries the zero (x_2,λ_2) of G_T. Let $T' := \Phi(T)$ be the triangulation induced by Φ and let $\sigma_1' := \Phi(\sigma_s)$. Then σ_1' is a completely labelled simplex for F with respect to T'_{λ_1}. Moreover, one has that $\text{mesh}T'_{\lambda_k} = r\ \text{mesh}T_{\lambda_k}$. Now one can treat the problem $F(x,\lambda) = 0$ with respect to T' using σ_1' as a start for the pivoting scheme. Additionally, one can shrink or enlarge the 'size' of T' in λ-direction. Thus, (3.7) gives rise to an easy implementable device to change the meshsize of a triangulation within a simplicial algorithm either to

- increase precision $(r < 1)$ or
- accelerate the pivoting scheme $(r > 1)$.

This device could also be interpreted as a "virtual" triangulation of R^{N+1} which corresponds to an expansion or compression of the original triangulation (see figure 14). For many purposes our technique seems to be an alternative to the wellknown refining triangulations K_3 (Eaves, Saigal) or J_3, especially, because here one can use the easy implementable pivoting rules for a K-type triangulation which are given for example in [A-G_1].

4. NUMERICAL EXPERIENCE

The simplicial path following algorithm and the perturbation techniques of sections 3 and 4 have been implemented in FORTRAN-code using K-type triangulations with a variable meshsize according to 3. We discuss two hard numerical problems which have been selected from a large number of successful applications of the package to various problems in nonlinear numerical analysis.

(4.1) A NONLINEAR DIFFERENTIAL DELAY EQUATION

$$\text{DDE}\quad\begin{cases}\dot{u}(t) = -\lambda f(u(t-1)), & t \geq 0, \quad \lambda \in R_+\\ u(t) = \varphi(t), & -1 \leq t \leq 0\\ \varphi \in C[-1,0], & f(x) = x(1+x^8)^{-1}\ .\end{cases}$$

PROBLEM: Do there exist periodic solutions and what is the global picture of periodic solutions if λ varies?

We formulate an operator equation: Let $C_+ = \{\varphi \in C[-1,0]:\ \varphi(-1) = 0$ and φ is monotonic increasing$\}$. Observe that for any $\varphi \in C_+$ DDE has a unique solution $u(\varphi,\lambda;t)$. We define an operator $S: C_+ \times R_+ \to C_+$ (Poincaré-map) by setting

$$(4.2)\qquad S_\lambda(\varphi)(t) = \begin{cases}-u(\varphi,\lambda;z_1+t+1), & \text{if } z_1 \text{ exists}\\ 0 & \text{, else;}\end{cases}$$

where z_k denotes the k-th zero of $u(\varphi,\lambda;t)$ (see figure 15). Since DDE is autonomous and f is odd one immediately concludes that any fixed point $S_\lambda(\varphi) = \varphi$ yields a periodic solution $u(\varphi,\lambda;t)$. Furthermore, by definition of S_λ one has trivial solutions $S_\lambda(0) = 0$ and one can show that S_λ is continuous (see [N]). It is not known, however, whether S_λ has any differentiability properties which could be exploited for classical numerical procedures. Let S_λ^k denote the k-th iterate of S_λ and observe that

$S_\lambda(\varphi) = \varphi$ implies that $S_\lambda^k(\varphi) = \varphi$ for all $k \in N$.

However, there might be solutions of $S_\lambda^k(\varphi) = \varphi$ which are not solutions of $S_\lambda^i(\varphi) = \varphi$ $(i < k)$.

4.3 CONJECTURE (R.D. Nussbaum, 1977)

There are $\lambda_o = \pi/2 < \lambda_1 < \lambda_2 < \dots$ such that S_λ has bifurcation in λ_o, S_λ^2 has bifurcations in λ_1 and λ_o, S_λ^4 has bifurcations in λ_2, λ_1 and λ_o, e.t.c.

Our aim was to check this conjecture. To obtain a suitable problem $F: R^{N+1} \to R^N$ we have discretized the initial values $\varphi \in C_+$ on a uniform grid $t_i = i/_N - 1$, $i=1,\dots,N$, i.e. we define $x = (x_1,\dots,x_N)$ with $x_i = \varphi(t_i)$. Then $u(\varphi,\lambda;t)$ has been approximated by an integration of DDE based on the grid t_i using a 4th order Newton-Cotes method (Milne-rule) and $S_\lambda^k(\varphi)$ has been approximated by linear interpolation of the discretized solution $u(\varphi,\lambda;t)$. If we denote this operation on $x \in R^N$ again by S_λ^k then we can define

$$F_k: R^N \times R \to R^N \quad \text{by}$$

$$F_k(x,\lambda) := x - S_\lambda^k(x).$$

(4.4) A NONLINEAR ELLIPTIC EIGENVALUE PROBLEM

$$\text{PDE} \quad \begin{cases} -\Delta u = \lambda f(u), & \lambda \in R_+, \quad \Omega = [0,\pi]^2 \\ u_{|\partial\Omega} = 0 \end{cases}$$

$$f(u) = \exp(u), \quad f(u) = u^3.$$

We have used finite difference approximations on a uniform grid of $N = M^2$ internal meshpoints in Ω and thus obtained a problem of the form

$$F: R^N \times R \to R^N \quad \text{with}$$

$$F(x,\lambda) = Ax - \lambda f(x),$$

where A denotes the approximation of $-\Delta$ respecting the boundary conditions (see [I-K]). Our interest was especially in the nonlinearity $f(u) = u^3$ for which one knows that PDE has infinitely many solutions [A-R].

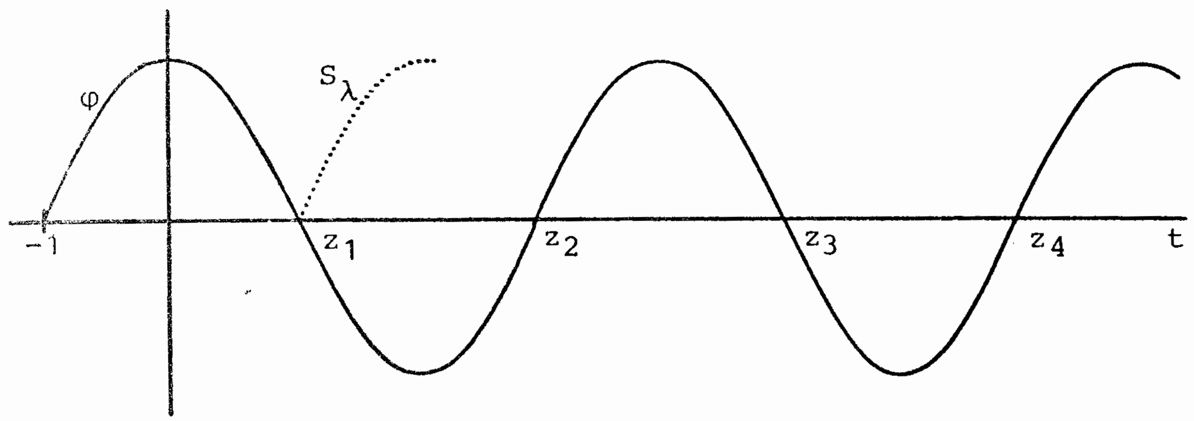


Figure 15.

(4.5) RESULTS

Problem (4.1) has been discretized on N = 20 meshpoints on [-1,0] and the meshsize of T was mesh(T_λ) = 10^{-3}. The average error e(x) := $|S_\lambda^k(x)-x|$ (| | = max-norm) was e(x) ~ 10^{-5}. In figure 17, we give the bifurcation diagrams for S_λ, S_λ^2 and S_λ^4 as well as the periodic solutions for λ = 2.7 and λ = 3.75.

Problem (4.4) has been discretized on N = 25 and N = 64 meshpoints and the meshsize of T was mesh(T_λ) = 10^{-2}. The average error e(x) = $|F(x,\lambda)|$ was e(x) ~ 10^{-4}. For f(u)=exp(u) we have found the wellknown caplike positive solutions (cf. [M-S]). For f(u) = u^3 we have found a large number of solutions as a result of perturbations of type G ≡ c ∈ $R^N \setminus \{0\}$ and G = P ∘ F, P ∈ GL_-. Figure 16 shows computer plots which have been made on the basis of our computations.

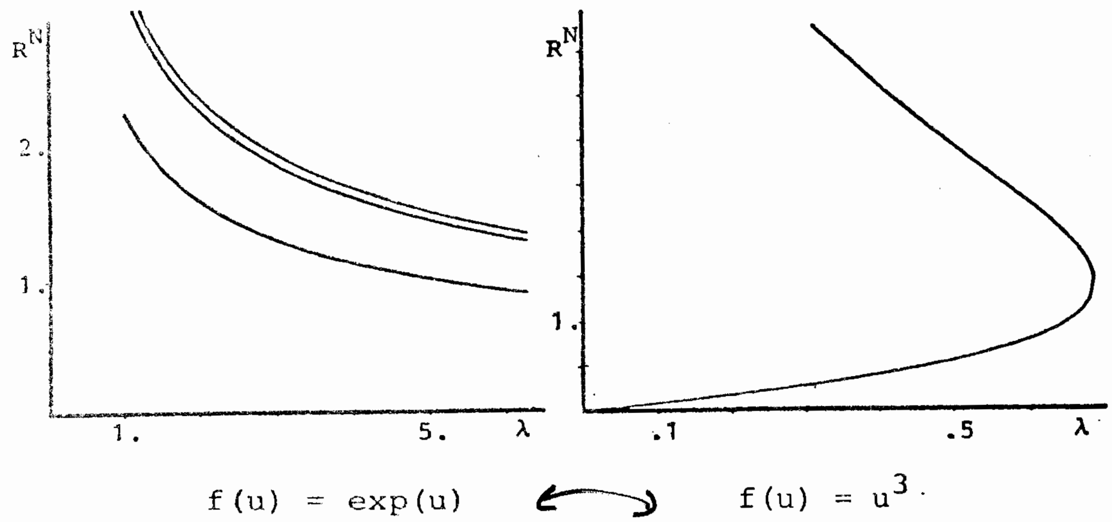


Figure 16.

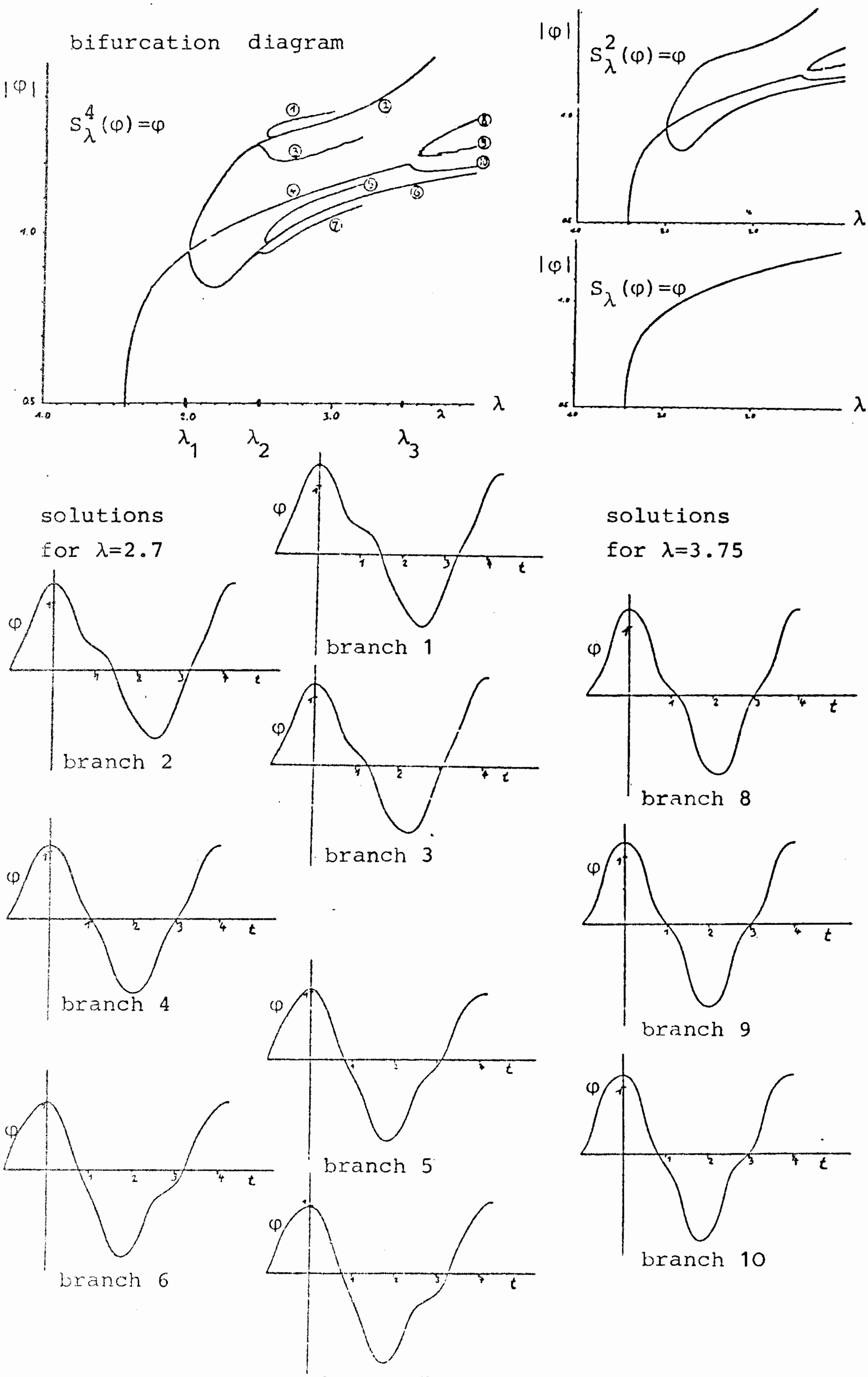

bifurcation diagram
|φ|
$S_\lambda^4(\varphi)=\varphi$
λ
λ_1
λ_2
λ_3
|φ|
$S_\lambda^2(\varphi)=\varphi$
λ
|φ|
$S_\lambda(\varphi)=\varphi$
λ
solutions
for λ=2.7
branch 1
branch 2
branch 3
branch 4
branch 5
branch 6
branch 7
solutions
for λ=3.75
branch 8
branch 9
branch 10

Figure 17.

(4.6) SOME DETAILS ABOUT THE PERTURBATIONS FOR DDE

As figure 17 suggests the DDE (4.1) has many distinct periodic solutions which are 'very close' to each other. Thus, in order to resolve the involved global structure of bifurcating solutions one has to use a relatively small meshsize for T (mesh(T) $\sim 10^{-3}$) and at the same time one has to discretize the initial values $\varphi \in C_+$ on a relatively large number of meshpoints ($N \sim 20$) in order to guarantee that the almost idential structures of periodic solutions can be distinguished.

In view of this the function evaluations F, F_2 and especially F_4, which are necessary in each step of the pivoting scheme, would automatically lead to a computing time beyond any reasonable scale (> 24h). It is due to the very effective acceleration techniques introduced in section 3 that problem (4.1) could be solved in a reasonable amount of time.

Finally, we discuss some selected perturbations which were used to obtain the computer plots in figure 17 and which justify Nussbaum's conjecture (4.3). All perturbations H_T are subject to (2.23).

(4.7) HOW TO FIND A COMPLETELY LABELLED SIMPLEX FOR $F_2(x,\lambda) = x - S_\lambda^2(x) = 0$ NEAR $x = 0$ AND $\lambda = 1$:

We homotop F_2 with the trivial problem $G(x,\lambda) \equiv x$ which possesses an easy detectable completely labelled simplex σ_o. More precisely, we choose the following perturbation (T denotes a K-type triangulation of R^{N+1}):

$$\begin{cases} \Gamma_- = R^N \times (-\infty, 1] \\ \Gamma_+ = \{(x,\lambda) \in R^{N+1} : (x,\lambda) \in \sigma^{N+1} \in T \text{ and } \sigma^{N+1} \cap \Gamma_- = \emptyset\} \\ F_2(x,\lambda) = x - S_\lambda^2(x), \\ G(x,\lambda) \equiv x, \\ H_1 \text{ the extension of } F_2 \text{ and } G \end{cases}$$

$\sigma_o \in T \cap R^N \times \{1\}$ generates a chain $ch^T_{H_1}(\sigma_o)$ which enters Γ_+ and, thus, provides completely labelled simplices for F_2.

Moreover, this chain bifurcates of from the trivial solutions $x = 0$ at $\lambda = \pi/2$ subject to (1.8).

Unfortunately, however, the nontrivial solutions corresponding to $ch^T_{H_1}(\sigma_0)$ are negative (due to the oddness of f one observes that if u(t) is a solution of DDE then also -u(t) is a solution) (see figure 18).

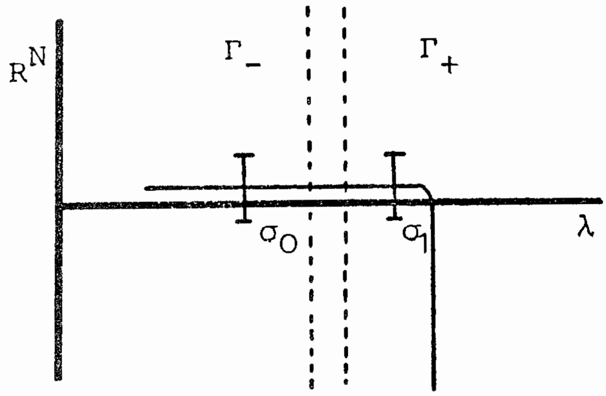


Figure 18.

To obtain the corresponding positive solutions we describe a second perturbation:

(4.8) HOW TO OBTAIN POSITIVE SOLUTIONS FROM NEGATIVE SOLUTIONS NEAR THE FIRST BIFURCATION:

We choose

$$\begin{cases} \Gamma_- = \{(x,\lambda): 1.4 \le \lambda \ , \ |x| = \max|x_i| \le 2\cdot \text{mesh}(T)\} \\ \Gamma_+ = \{(x,\lambda): (x,\lambda) \in \sigma^{N+1} \in T \text{ and } \sigma^{N+1} \cap \Gamma_- = \emptyset\} \\ F_2(x,\lambda) = x - S^2_\lambda(x) \\ G(x,\lambda) \equiv (0,\ldots,0,-1) \\ H_2 \text{ the extension of } F_2 \text{ and } G \ . \end{cases}$$

Subject to the first perturbation (4.7) we have a completely labelled simplex σ_1 (see figure 18) for F_2 which approximates $x = 0$ before the first bifurcation point $\lambda_0 = \pi/2$ With σ_1 we start the second perturbation (4.8) that is we consider the chain $ch^T_{H_2}(\sigma_1)$ which is plotted in figure 19.

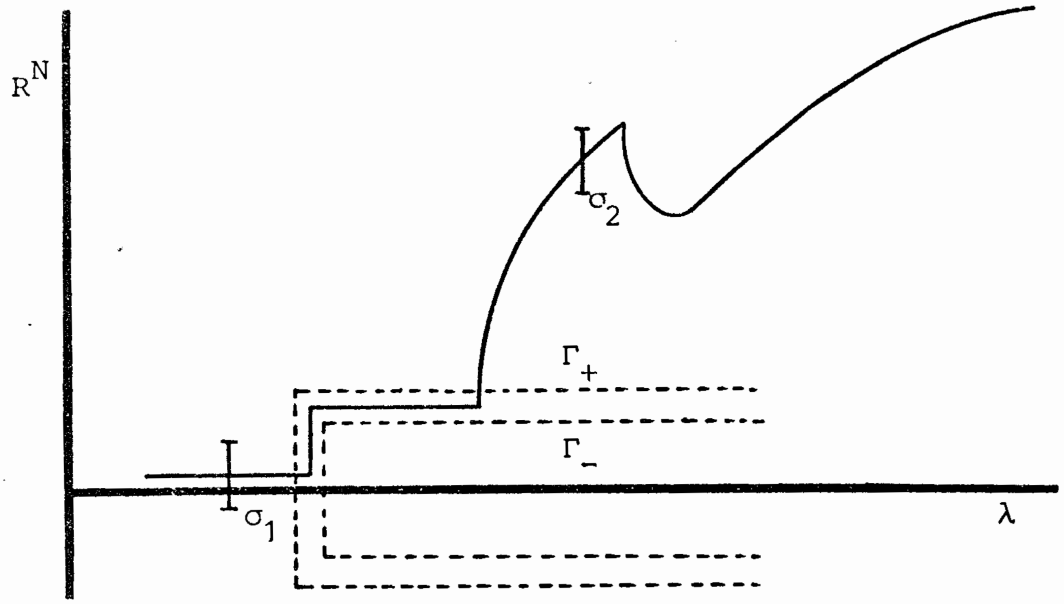


Figure 19.

We observe that $ch^T_{H_2}(\sigma_1)$ enters Γ_a, bifurcates in Γ_a from the trivial solutions and then again bifurcates in Γ_a to leave Γ_a and enter Γ_+, thus, approximating solutions of $F_2(x,\lambda) = 0$. These solutions are positive solutions approximating solutions of DDE. Pursueing $ch^T_{H_2}(\sigma_1)$ in Γ_+ we pass a second bifurcation point at λ_1 subject to (1.8) (see figure 19).

(4.9) HOW TO OBTAIN ALL BRANCHES AT THE SECONDARY BIFURCATION POINT λ_1:

We choose the perturbation

$$\begin{cases} \Gamma_- = \{(x,\lambda):\ \lambda_1 < 2.1 \leq \lambda\} \\ \Gamma_+ = \{(x,\lambda):\ (x,\lambda) \in \sigma^{N+1} \in T \text{ and } \sigma^{N+1} \cap \Gamma_- = \emptyset\} \\ F_2(x,\lambda) = x - S^2_\lambda(x) \\ G(x,\lambda) = (0,\ldots,0,-1) \\ H_3 \text{ the extension of } F_2 \text{ and } G . \end{cases}$$

Let $\sigma_2 \in ch^T_{H_2}(\sigma_1)$ be given as in figure 19 which is a completely labelled simplex approximating a zero of F_2 before the second bifurcation point λ_1. We pursue the chain $ch^T_{H_3}(\sigma_2)$

defined by the perturbation H_3 and obtain a chain visualized in figure 20:

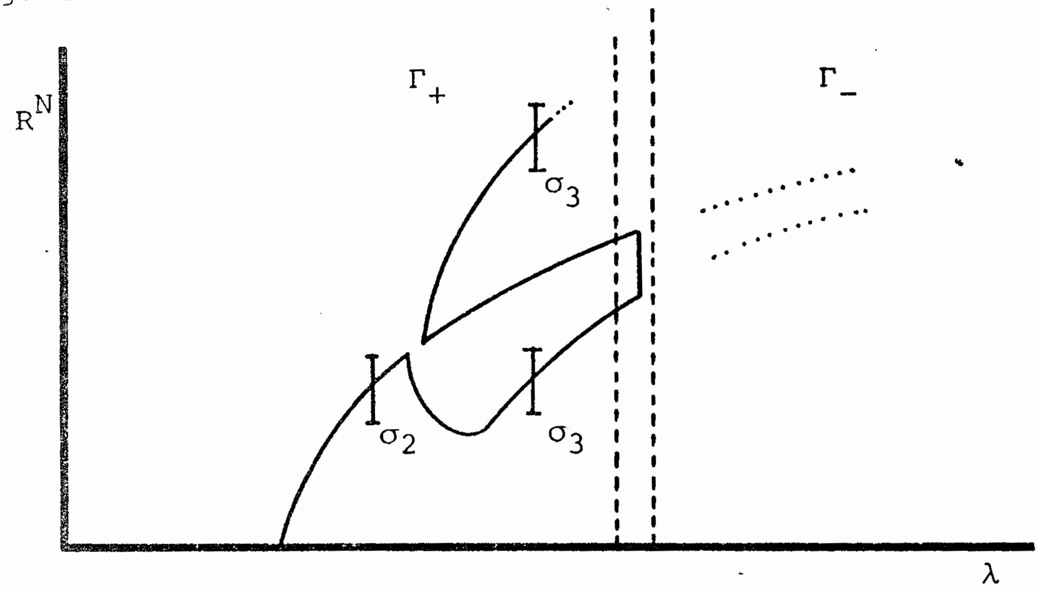


Figure 20.

Implementing the same technique one obtains all branches at the tertiary bifurcation point $\lambda_3 \sim 3.5$ (see figure 17, branch 4 and 9).

(4.10) <u>HOW TO OBTAIN SOLUTIONS OF</u> $S_\lambda^4(x) = x$ <u>WHICH ARE NOT SOLUTIONS OF</u> $S_\lambda^2(x) = x$:

We choose

$$\begin{cases} \Gamma_- = \{(x,\lambda):\ \lambda \le 2.4 < \lambda_2\} \\ \Gamma_+ = \{(x,\lambda):\ (x,\lambda) \in \sigma^{N+1} \in T \text{ and } \sigma^{N+1} \cap \Gamma_- = \emptyset\} \\ F_4(x,\lambda) = x - S_\lambda^4(x) \\ G(x,\lambda) = x - S_\lambda^2(x) \\ H_4 \text{ the extension of } F_4 \text{ and } G\ . \end{cases}$$

According to the above discussion we may assume that we have computed the completely labelled simplices for G in Γ_-. Each chain of completely labelled simplices for G (e.g. $ch_{H4}^T(\sigma_3)$) may be pursued into Γ_+, where at first these chains approximate solutions of $F_4(x,\lambda) = 0$, which are also solutions to $F_2(x,\lambda) = 0$, and then these chains bifurcate at $\lambda_2 \sim 2.5$ to approximate solutions which do not solve $F_2(x,\lambda) = 0$ (see figure 21).

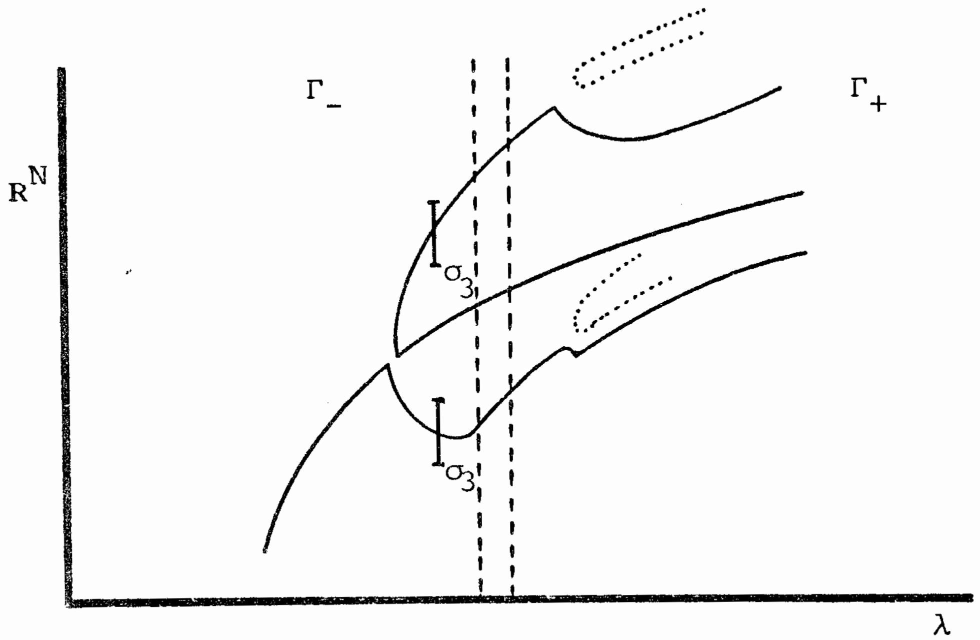


Figure 21.

A slight modification of the last perturbation H_4 provides the other continua of solutions for $S_\lambda^4(x) = x$: We choose

$$\begin{cases} \Gamma_- = \{(x,\lambda): \quad \lambda \leq 2.7\} \\ \Gamma_+ = \{(x,\lambda): (x,\lambda) \in \sigma^{N+1} \in T \text{ and } \sigma^{N+1} \cap \Gamma_- = \emptyset \} \\ \tilde{F}_4(x,\lambda) = P \circ F_4(x,\lambda), \qquad P = \begin{pmatrix} -1 & & & 0 \\ & +1 & & \\ & & \ddots & \\ 0 & & & +1 \end{pmatrix} . \\ G(x,\lambda) = F_2(x,\lambda) \\ H_5 \text{ the extension of } \tilde{F}_4 \text{ and } G . \end{cases}$$

Observe that $\tilde{F}_4^{-1}(0) = F_4^{-1}(0)$, however, the orientation of completely labelled simplices under F_4 is opposite to the orientation of corresponding completely labelled simplices under $\tilde{F}_4$. The perturbation H_5 may be viewed at as a combination of techniques discussed in section 2 separately. Observe that the solutions on branches 2 and 6 have opposite degree with respect to F_2 and F_4. The purpose of perturbation H_5 is to give these solutions the same degree in Γ_+ and Γ_- and, thus, make them accessible by the algorithm.

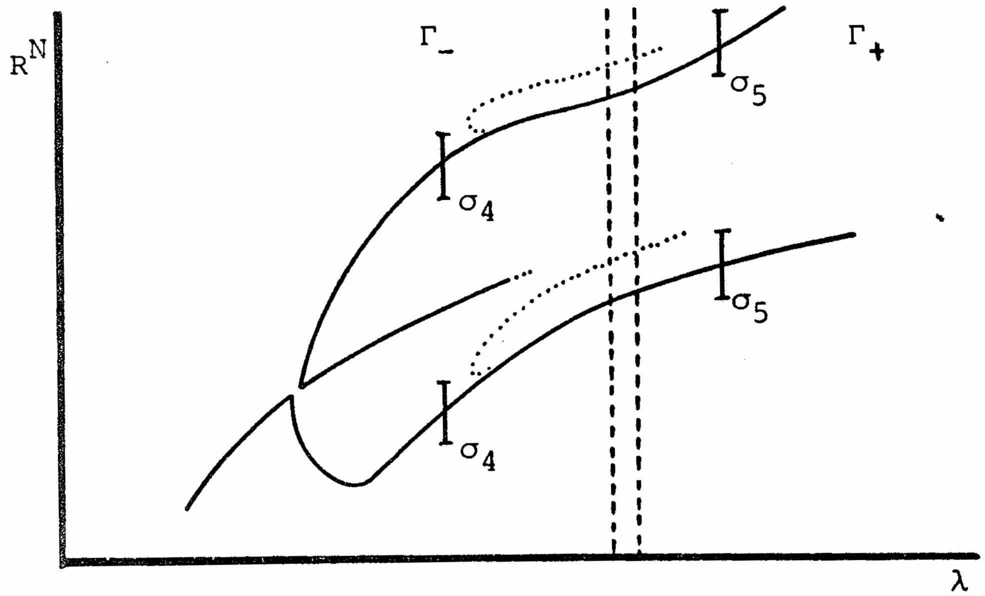


Figure 22.

Once, a chain (e.g. $ch^T_{H5}(\sigma_4)$) has entered Γ_+ (e.g. $\sigma_5 \in ch^T_{H5}(\sigma_4)$) then one may switch off the perturbation H_5 and may obtain the dotted part (e.g. via $ch^T_{F4}(\sigma_5)$) according to (1.8) as a bifurcation branch.

(4.11) CONCLUDING REMARKS

Recently (see [Sch-S]) it has been shown that one may obtain bifurcation results for equations

$$\text{ODE} \begin{cases} -u'' = \lambda u + f(u) \\ u(0) = u(1) = 0 \end{cases}$$

even if the nonlinearity is not differentiable at 0.

We have experimented with two nonlinearities:

$$f_1(u) = |u|$$

$$f_2(u) = u \cdot \sin(^1/_u) .$$

For f_2 one has the typical phenomenon that bifurcation (from trivial solutions) takes place in an entire interval (figure 23(a) shows what one would expect from [Sch-S] and figure 23(b) is a computer plot for solutions of ODE for the nonlinearity f_2).

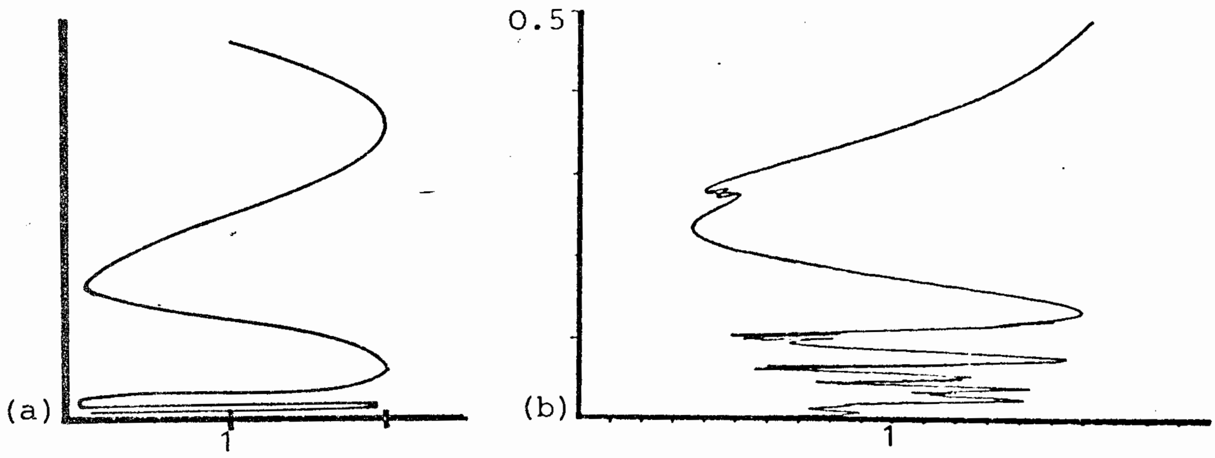


Figure 23.

In both cases, f_1 and f_2, we have computed bifurcation points (respectively bifurcation intervals) using a simplicial path following algorithm subject to (1.8) and the perturbation techniques discussed in sections 2 and 3.

These and examples (4.1) and (4.4) seem to indicate that simplicial path following algorithms may have a value in the global numerical study of difficult nonlinear eigenvalue and bifurcation problems.

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