Normalized and xPower to Generate Pacing Strategies in Road Cycling

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Abstract: Normalized power and xPower have become standard measures for metabolic demand with road cycling. In contrast to average power these quantities account for the additional exertion resulting from time-variations of the pedaling power. In this contribution, we develop a method to use a slightly modified xPower to generate pacing strategies for road cycling time trials, where determining the optimal variations of the pedaling power is the ultimate goal. We tackle severe numerical issues related to singular arcs in optimal control, derive the necessary condition and use the Chebfun software system with its essential and unique capability to compute with Fréchet derivatives automatically.

Keywords: Pacing Strategy, Road Cycling, Optimal Control, Normalized Power, xPower

1. INTRODUCTION

Normalized power, Allen and Coggan [2005], and xPower are essential quantities in popular training analysis software such as TrainingPeaks and Golden Cheetah. In contrast to average power, these quantities are intended to account for the additional metabolic demand due to the variability of pedaling power during a ride.

The 3-Parameter Critical Power model (3PCPM), Morton [1996], has been used in conjunction with a mechanical bicycling model, Martin et al. [1998] to compute individual minimum-time pacing strategies for cycling time trials on a track with given slope profile, Dahmen [2012]. However, the 3PCPM has been validated and is widely used to describe how long a cyclist can hold an arbitrary but time-constant power only. The generalization to variable power required for the optimization is a venturous and unrealistic assumption.

If normalized power and xPower are beneficial replacements for average power when describing the metabolic effort of a ride, then they should be beneficial as replacement for pedaling power in the 3PCPM when generating optimal strategies.

In Dahmen [2012], it was pointed out that using the 3PCPM for the optimization leads to singular optimal control problems and a regularization of the model was necessary to obtain a solution using general-purpose optimal control software (GPOPS). These problems remain in this contribution. Therefore, we manually derive the necessary condition for the optimal control problem. Then we use a customized version of the Chebfun software system, Driscoll et al. [2014]. Its unique feature of automatic Fréchet differentiation, adopted from Chebfun, allows us to overcome these numerical issues properly.

2. PROBLEM DEFINITION

The kinetic energy $e_{\text{kin}} \in \mathbb{L}^2([0,t_f])$ and the remaining anaerobic resources $e_{\text{an}} \in \mathbb{L}^2$ are functions of time $t \in [0,t_f]$, with $t_f \in \mathbb{R}$ being the final time. We put the variables into the vector $\boldsymbol{y} = (e_{\text{kin}}, e_{\text{an}}, t_f)^{\mathrm{T}}$ and seek

$$\hat{\boldsymbol{y}} = \arg\min_{\boldsymbol{y}\in\mathcal{D}} t_f$$

subject to the domain subspace $\mathcal{D}$. This subspace is defined by the equality constraints $\mathbf{L}(\hat{\boldsymbol{y}})(t) = 0$ with

$$\mathbf{L} = \begin{pmatrix} \dot{e}_{\text{kin}} + \dot{e}_{\text{an}} + \mathrm{P}_{\text{res}}(x, e_{\text{kin}}) - \mathrm{P}_{\mathrm{c}} \\ [e_{\text{kin}}]^0 \\ [e_{\text{an}}]^0 - \mathrm{E}_{\text{an}} \\ \int_0^{t_f} \sqrt{\frac{2e_{\text{kin}}}{\mathrm{m}}}\,\mathrm{d}t - \mathrm{x}_f \end{pmatrix}, \tag{1}$$

as well as the inequality path constraints $\mathbf{N}(\hat{\boldsymbol{y}})(t) \geq \mathbf{0}$ with

$$\mathbf{N} = \begin{pmatrix} e_{\text{an}} \\ \dot{e}_{\text{an}} + \mathrm{P}_{\max} - \mathrm{P}_{\mathrm{c}} \end{pmatrix}. \tag{2}$$

Note that L_1, the first component of $\mathbf{L}$ represents a path equality constraint whereas the last three components represent functionals.

We parametrize the problem using the kinetic energy e_{kin} as the variable function instead of the speed $v = \sqrt{2e_{\text{kin}}/\mathrm{m}}$, thus avoiding a singularity that arises when $\dot{v} = (P - P_{res})/(mv)$ is undefined for $v = 0$.

The differential constraint L_1 comprises both the mechanical model $\dot{e}_{\text{kin}} = P - P_{\text{res}}$ and the physiological model $\dot{e}_{\text{an}} = \mathrm{P}_{\mathrm{c}} - P$. Thereby, we have eliminated the pedaling power P.

The mechanical model stems from Martin et al. [1998]. We have $\mathrm{P}_{\text{res}} = F_{\text{res}} v$ with the resistance force

$$F_{\text{res}} = \mathrm{mgh}'(x) + \mathrm{k}_1 + \mathrm{k}_2 v + \mathrm{k}_3 v^2 . \tag{3}$$

The constants m, g, and $\mathrm{h}(x)$ represent the mass of the cyclist and the bike, gravity, and the slope profile, respectively. The fixed parameters in $\mathbf{k}$ are coefficients quantifying friction and areal drag.

The physiological model $\dot{e}_{\text{an}} = \mathrm{P}_{\mathrm{c}} - P$ originates from Morton [1996]. It quantifies the anaerobic resources e_{an} decreasing at a rate equal to the amount by which the pedaling power P exceeds the critical power P_{c} .

The functional constraints L_2–L_4 enforce zero initial kinetic energy, initial anaerobic resources equal to E_{an} (characteristic for the cyclist), and that the total distance covered at the final time be equal to the length of the track, x_f.

Furthermore, $\mathbf{N}$ ensures nonnegative anaerobic resources throughout the ride and limits the pedaling power $P = P_c - \dot{e}_{an}$ by some fixed maximum power, P_{max}.

To compute normalized power $N \in \mathbb{R}$ referring to any section $t \in [t_a, t_b]$ of a ride, the τ-moving average $Q_N \in \mathbb{L}^2$ of the power function with $\tau = 30\,\mathrm{s}$ is raised to the fourth power and then the fourth root of the average of the result is defined as normalized power:

$$N = \sqrt[4]{\int_{t_a}^{t_b} \frac{Q_N^4(t)}{t_b - t_a}\,\mathrm{d}t}\,. \tag{4}$$

With the modification xPower $X \in \mathbb{R}$, the average $Q_X(t)$ is exponentially weighted with the time constant τ. Mathematically, xPower has the advantage that $\dot{Q}_X = P - Q_X$ applies.

We modify xPower further by changing the order of averaging and raising to the fourth power to obtain $\dot{Q}(t) = P^4(t) - Q(t)$ with $Q \in \mathbb{L}^2$ being instantaneous modified xPower. We add Q to the variable vector $\boldsymbol{y} = (e_{kin}, e_{an}, Q, t_f)^{\mathrm{T}}$ and modify the differential equality constraint operator:

$$L_1 = \dot{Q} - \dot{e}_{an} + P_c - (\dot{e}_{kin} + P_{res}(e_{kin}))^4\,, \tag{5}$$

Moreover, we have to add the functional

$$L_5 = [Q]^0 \tag{6}$$

to ensure that the cyclist is rested at the beginning.

3. NECESSARY CONDITION AND IMPLEMENTATION

Note that the cost functional can be expressed as $t_f = \int_0^{t_f} \mathrm{F}\,\mathrm{d}t$ with the trivial running cost $\mathrm{F} = 1$. Generally, if $\hat{\boldsymbol{y}}$ is optimal, then the direction of steepest ascent of the running cost $\mathrm{F}(\hat{\boldsymbol{y}})$ must necessarily be perpendicular to the hyper-surfaces $\mathbf{L}(\hat{\boldsymbol{y}}) = 0$. In other words the direction of steepest ascent of F and each component of $\mathbf{L}$ must be co-linear. In addition, for each point in time, the inequality constraints can either be active or inactive, i.e., $\mathbf{N}(\hat{\boldsymbol{y}})(t) = 0$ or $\mathbf{N}(\hat{\boldsymbol{y}})(t) > 0$. If a component of $\mathbf{N}$ is active, then the direction of steepest ascent of F must be co-linear with the corresponding component of $\mathbf{N}$, too.

We introduce the adjoint variables $\boldsymbol{\lambda}$ with $\lambda_1 \in \mathbb{L}^2$ and $(\lambda_1, \lambda_2, \lambda_3)^{\mathrm{T}} \in \mathbb{R}^3$ as well as $\boldsymbol{\nu} \in (\mathbb{L}^2)^2$ and form the augmented cost functional $\int_0^{t_f} \mathrm{H}\,\mathrm{d}t$ with the Hamiltonian operator

$$\mathrm{H} = \mathrm{F} + \boldsymbol{\lambda}^{\mathbf{T}}\mathbf{L} + \boldsymbol{\nu}^{\mathbf{T}}\mathbf{N} \tag{7}$$

The Hamiltonian operator H comprises all information of our problem. Note that for the functional derivatives $\delta\mathrm{H}/\boldsymbol{\delta\lambda} = \mathbf{L}$ and $\delta\mathrm{H}/\boldsymbol{\delta\nu} = \mathbf{N}$ apply by construction. Thus, $\mathbf{L}$ and $\mathbf{N}$ determine the sensitivity of H with respect to their corresponding adjoint variable.

Similarly the adjoint operators $\delta\mathrm{H}/\boldsymbol{\delta y} = ((\mathbf{L}, \mathbf{N})^{\mathrm{T}})^*$ determine the sensitivity of H with respect to the primal variables.

If a component N_i is inactive on some interval, then $\nu_i = 0$ on that same interval, otherwise $\nu_i > 0$. In other words $N_i(\hat{\boldsymbol{y}})$ and ν_i are complementary $N_i(\hat{\boldsymbol{y}}) \perp \nu_i$. We now seek the dual solution that in addition satisfies the adjoint constraints $\delta\mathrm{H}/\boldsymbol{\delta}\hat{\boldsymbol{y}}\,(t) = \mathbf{0}$.

Generally, the dual problem represents a multipoint boundary value problem with split boundary conditions. In contrast to our original problem, the dual problem has the property that it is complete. I.e., if split at the junctions where either N_1 or N_2 change from active to inactive or vice versa, on each interval between the junctions, it is guaranteed that the number of functional constraints to determine the solution is correct.

Discretizing a complete linear boundary value problems using spectral collocation results in a well-posed linear equations system that has a unique solution. Our nonlinear problem may be solved by a sequence of Newton-Kantorovich iterations. This approach is taken by the Chebfun software system, Driscoll et al. [2014], which we make use of. Its unique feature of automatic Fréchet differentiation, is essential for the computations.

4. CONCLUSIONS

We emphasize that this approach is still work in progress since we strive to integrate our manual derivation into the automatic differentiation of the Chebfun system. Moreover, our Newton-Kantorovich iteration is not yet as robust as Chebfun's original implementation since we have not yet integrated the step size control.

Clearly, it remains to implement the algorithm for the original definition of xPower and Normalized Power. Those approaches are more complex since they involve delay differential equation. Eventually, we plan to perform tests on our bicycling simulator to assess if using any of the variants of normalized and xPower is beneficial for the generation of minimum-time pacing strategies in practice.

Our bicycle simulator is already capable of showing the open-loop optimal strategy to the athlete, but as we expect that an athlete cannot always follow exactly a strategy that is at the limit of his physical capacity, a closed loop feedback control based on model predictive control is desirable for practical use in future.

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