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Computing a Field of Optimal Pacing Strategies for Cycling Time Trials

Introduction, Related Work, and Target Application

Optimal pacing strategies for cycling time trials have been treated in literature in numerous publications, e.g. Gordon (2005). Generally, the minimum time pacing is sought subject to a mechanical model and a physiological model. The mechanical model (Martin, 1998) comprises the pedaling power P and forces for gravity on a cycling track of varying slope, rolling resistance, aerodynamic drag, frictional losses in wheel bearings, and inertia. The physiological model defines the capability of a cyclist to perform a pedaling power.
Both models have the form of a set of non-linear differential equations and potentially additional pedal power constraints:

$$\begin{aligned} \dot{v} &= f(P, x, v) \\ \dot{\boldsymbol{e}} &= \boldsymbol{g}(P) \end{aligned} \quad \text{and} \quad P \leq P_m(\boldsymbol{e})$$

where x, v, $\boldsymbol{e}$ denote state variables for the distance, speed, and quantities defining the physiological state of the cyclist. The functions f and g represent the mechanical and the physiological model, respectively, whereas P_m is the maximum affordable pedaling power.
Recently, Dahmen (2012) discussed numerical solutions for the minimum time pacing strategy resulting in an optimal power control P and an optimal trajectory in the state space as functions the distance x using state-of-the-art, general purpose and open-source optimal control software (Rao, 2010) for three different physiological models.
However, in particular the physiological model is generally prone to many disturbing influences during training and competition. In practice, it is impossible to follow a pre-computed minimum time pacing strategy exactly.
Therefore, in this contribution, we extend the approach such that we compute a field of minimum time pacing strategies, i.e., an optimal closed-loop pedal power law as a function not only of the distance but also of the state variables v and e in a neighborhood R of the previously computed trajectory. The optimal solution is stored for discrete states in R. Then the optimal pedal power can be interpolated efficiently for any state in R and thus be presented as feedback to the cyclist during training and in competition.

Computing the field of optimal pacing strategies

In recent years, mature open-source software packages for solving general optimal control problems have become available (Rao 2010, Becerra 2010). Commonly, direct transcription using pseudospectral methods is employed and recommended in

particular if only at least piecewise smooth functions and simple domains are involved. These conditions are met by the minimum time pacing strategy problem at hand and several versions of this problem have been solved successfully in Dahmen (2012).
Methods for the computation of the field of minimum time pacing strategies are derived from Bellman's principle of optimality. For the same reasons as above, we chose a direct method, specifically dynamic programming.
Furthermore, we decide to use pseudospectral collocation methods for discretization in order to minimize the grid density and thus tackle the curse of dimensionality, which generally arises with dynamic programming.
More precisely, we use the following Algorithm to compute the optimal feedback pedal power, which is depicted in Figure 1.

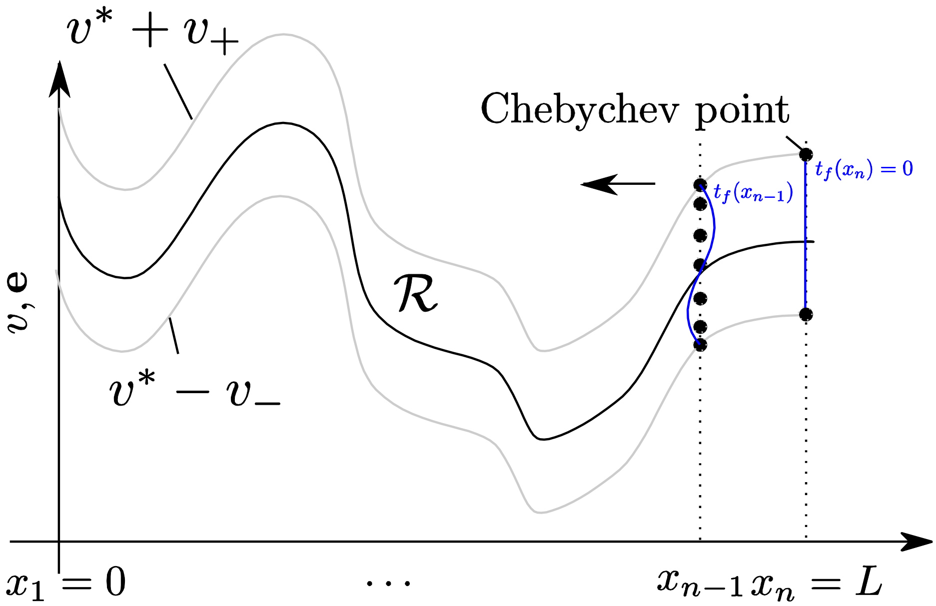

Fig 1: Computing the field of optimal pacing strategies using dynamic programming on pseudospectral grids.

1. Compute an optimal pacing strategy as described in Dahmen (2012). The solution is given on n discrete distances x , with $1 \leq i \leq n$ as a result of mesh refinement.
2. Define a box-shaped region R in the state space as the neighborhood of the optimal pacing trajectory for which we want to obtain the solution:
$$R\colon \begin{pmatrix} 0 \\ \max(v^* - v_-, 0) \\ \max(\boldsymbol{e}^* + \boldsymbol{e}_-, 0) \end{pmatrix} \leq \begin{pmatrix} x \\ v \\ \boldsymbol{e} \end{pmatrix} \leq \begin{pmatrix} L \\ v^* + v_+ \\ \min(E_{\max}, \boldsymbol{e}^* + \boldsymbol{e}_+) \end{pmatrix}.$$
3. For each point in R, we define the time to finish the course from this point $t_f(x_n, v, \boldsymbol{e})$ as the cost functional to be minimized. Trivially, at the finish line, $t_f(x_n, v, \boldsymbol{e}) \equiv t_f^*(x_n, v, \boldsymbol{e}) \equiv 0$. This constant function can be perfectly represented by the function values at the 4 Chebychev points in the corners of R at x_n.

4. Slightly, increase the number of Chebychev points in the plane x_{n-1}. Calculate the minimum time to finish the course and the optimal pedal power on these Chebychev points using the GPOPS software (Rao, 2010) with the sum of $t_f(x_n)$ and the time to reach x_n from x_{n-1} as the cost functional. The corresponding Chebychev expansion defines $t_f(x_{n-1}, v, \boldsymbol{e})$. Reduce the number of Chebychev points in x_{n-1} as far as the accurary criterion holds (Trefethen, 2011).
5. Repeat step 4 with n replaced by $n-1$ until $n = 2$.

Saving the values $P(x_i, v, \boldsymbol{e})$ and $t_f(x_i, v, \boldsymbol{e})$ at the Chebychev points in each plane x_i allows to compute the optimal pacing using barycentric interpolation with linear complexity within R, (Trefethen, 2012).

Results and Discussion

As an example, Figure 2 depicts the slope of a course as well as the optimal power P^* in the $e = e*$ -plane.

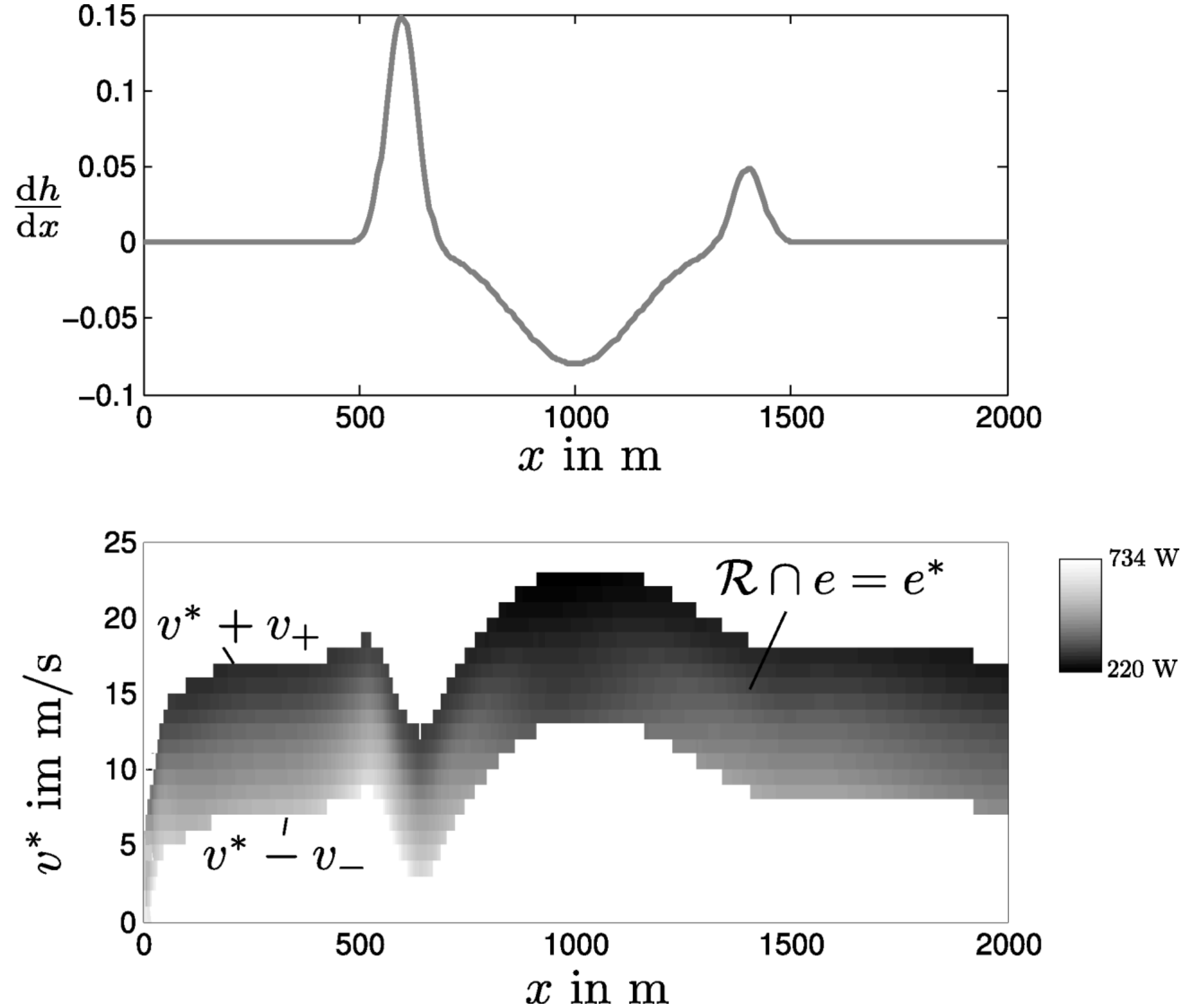

Fig. 2: Sample field of optimal pacing strategy.

The parameters for the mechanical and physiological models are taken from Dahmen (2012) which is itself based on Gordon (2005). It is clearly visible, that the optimal pedalling power increases if the cyclist moves to slow or if he faces a steep ascent. The proposed method is general and should be useful in many optimal control problems, where the computation of a field is desired.

References

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